

P. Chojedei "On the geometric version"

§1. Introduction

Notations: $p > 2$ prime, F/\mathbb{F}_p finite, $\bar{r}: G_{\mathbb{Q}_p} \rightarrow GL_2(F)$ continuous, $E/W(F)[\frac{1}{p}]$ finite, tot. unramified, $\theta = \theta_E$, π unif., (E suff. large)

$\tau: I_{\mathbb{Q}_p} \rightarrow GL_2(E)$ "inertial type": rep. with open kernel which extends to $G_{\mathbb{Q}_p}$.

Fix $a, b \in \mathbb{Z}$, $b \geq 0$ and character $\psi: G_{\mathbb{Q}_p} \rightarrow \mathcal{O}^\times$ s.t. $\overline{\psi \epsilon} = \det \bar{r}$

where ϵ is a p -adic cyclotomic character. Let:

$R^{\square, \psi}(a, b, \tau, \bar{r})$ (resp. $R_{cr}^{\square, \psi}(a, b, \tau, \bar{r})$) be the framed deformation \mathcal{O} -algebras which are universal for framed deformations of \bar{r} which have determinant $\psi \epsilon$, are pot. semistable (resp. pot. crystalline) with HT uts $(a, a+b+1)$ and inertial type τ .

Let $\sigma(\tau)$ (resp. $\sigma_{cr}(\tau)$) be fin. dim. ^{irred.} rep. of $GL_2(\mathbb{Z}_p)$ corr. to τ via Henniart's ~~inertial~~ inertial LLC, i.e. it is determined by the condition that

$\text{Hom}_{GL_2(\mathbb{Z}_p)}(\sigma(\tau), \pi) \neq 0 \Leftrightarrow LL(\pi)|_{I_{\mathbb{Q}_p}} \cong \tau$
(resp. $\text{Hom}_{GL_2(\mathbb{Z}_p)}(\sigma_{cr}(\tau), \pi) \neq 0 \Leftrightarrow LL(\pi)|_{I_{\mathbb{Q}_p}} \cong \tau$ and $N=0$)
where π is inf. dim. smooth abs. irred. rep. of $GL_2(\mathbb{Q}_p)$. (monodromy)

Set $\sigma(a, b, \tau) = \sigma(\tau) \otimes_E \det^a \text{Sym}^b E^2$ and similarly $\sigma_{cr}(a, b, \tau) = \sigma_{cr}(\tau) \otimes_E \det^a \text{Sym}^b E^2$

Let $L_{a, b, \tau}$ be $GL_2(\mathbb{Z}_p)$ -stable \mathcal{O} -lattice in $\sigma(a, b, \tau)$ (similarly for σ_{cr})

Write $\sigma_{m, n} = \det^m \otimes \text{Sym}^n F^2$ of $GL_2(F)$, $0 \leq m \leq p-2$, $0 \leq n \leq p-1$ so
 $(L_{a, b, \tau} \otimes_{\mathcal{O}} F)^{ss} \cong \bigoplus_{m, n} \sigma_{m, n}^{a, m, n}$; $(L_{cr, a, b, \tau} \otimes_{\mathcal{O}} F)^{ss} \cong \bigoplus_{m, n} \sigma_{m, n}^{cr, a, m, n}$ for some integers $a_{m, n}$

Now:

BM conj. $\exists \mu_{m, n}(\bar{r}) \in \mathbb{Z} \forall a, b, \tau$

$e(R^{\square, \psi}(a, b, \tau, \bar{r})/\pi) = \sum_{m, n} a_{m, n} \mu_{m, n}(\bar{r})$

$e(R_{cr}^{\square, \psi}(a, b, \tau, \bar{r})/\pi) = \sum_{m, n} a_{m, n}^{cr} \mu_{m, n}(\bar{r})$

Remark: if conj. is true $\forall a, b, \tau \Rightarrow \mu_{m, n}(\bar{r}) = e(R_{cr}^{\square, \psi}(m, n, 1, \bar{r})/\pi)$

Geometric version: $\forall_{m, n}$ as above $\exists c_{m, n}$ cycle dep. on m, n, \bar{r} s.t. $\forall a, b, \tau$:
 $Z(R^{\square, \psi}(a, b, \tau, \bar{r})/\pi) = \sum_{m, n} a_{m, n} c_{m, n}$ (and similarly for $R_{cr}^{\square, \psi}$)

Remark: if conj. is true $\forall a, b, \tau \Rightarrow c_{m, n} = Z(R_{cr}^{\square, \psi}(m, n, 1, \bar{r})/\pi)$

remark on cycles: $\sum_{\dim(x)=d} n_x \cdot x$ is a formal sum (on ^{locally} scheme X)

$\sum_{\dim(x)=d} n_x \cdot x$, $n_x \in \mathbb{Z}$; where $\dim(x)$ means that $\{x\}$ has dim d

if \mathcal{M} is a coh. sheaf on X whose support (i.e. $\mathbb{Z} \subset X$ does not cut out by annihilator ideal $I \subset \mathcal{O}_x$ of \mathcal{M}) has $\dim \leq d$. then define

$$Z_d(\mathcal{M}) = \sum_{\dim(x)=d} e(\mathcal{M}, x) \cdot x \quad \text{where } e(\mathcal{M}, x) = \begin{cases} 0 & \text{if } \mathcal{M}_x = 0 \\ \lg_{\mathbb{Z}} \mathcal{O}_{\mathbb{Z}, x}(\mathcal{M}_x) & \text{otherwise} \end{cases}$$

when the support is d -dimensional then

we write $Z_d(\mathcal{M}) = Z(\mathcal{M})$

by the results of Kish $(\mathbb{R}^D, \mathbb{P}/\dots)$ are eq-dim. for some d . (equidimensional)

↑ support

§2. Proof of geometric version:

one uses patching to show BH conj. \Rightarrow Geometric version:

start with proposition: $\exists F$ tot. real field, in which p is tot. decomposed and $\bar{\rho}: G_F \rightarrow GL_2(\mathbb{F})$ which is tot. odd, modular, unramified and $\forall v|p \bar{\rho}|_{G_{F_v}} \cong \bar{\rho}$ (moreover assume $\bar{\rho}(G_F) = GL_2(\mathbb{F})$ outside $v \nmid p$ technical cond.)

We will now sketch the argument with patching to which we return in future lectures.

For each $v|p$ (in F) let $R_v^{\square, \gamma}(\bar{\rho}|_{G_{F_v}})$ be the universal framed deformation \mathcal{O} -algebra for $\bar{\rho}|_{G_{F_v}}$ with determinant $\gamma \in \mathcal{O}^\times$ and define

$$R_{\infty} := \hat{\otimes}_{v|p, \mathcal{O}} R_v^{\square, \gamma}(\bar{\rho}|_{G_{F_v}})[[x_1, \dots, x_g]] \quad , \quad x_i \dots \text{ formal variables}$$

(where g depends on patching datum which we do not invoke)

For each $v|p$, choose $a_v, b_v \in \mathbb{Z}$, $b_v \geq 0$ and inertial type τ_v and let

$*$ be \ast (crystalline) or nothing. Assume that $2a_v + b_v$ does not depend on v .

Write $R_{\infty}^{\ast} := \hat{\otimes}_{v|p, \mathcal{O}} R_v^{\square, \gamma}(a_v, b_v, \tau_v, \bar{\rho}|_{G_{F_v}})[[x_1, \dots, x_g]]$
(cr or nothing.)

and $W_{\mathcal{O}} = \hat{\otimes}_{v|p} L_{a_v, b_v, \tau_v}^{\ast}$ - finite free \mathcal{O} -mod. with action of $\prod_{v|p} GL_2(\mathcal{O}_{F_v})$

$W_{\mathcal{O}}/\pi W_{\mathcal{O}}$ rep. over \mathbb{F} . Fix a Jordan-Hölder filtration of $W_{\mathcal{O}}/\pi W_{\mathcal{O}}$ by

$$\prod_{v|p} GL_2(\mathcal{O}_{F_v}) \text{ (sub) reps } 0 = L_0 \subset \dots \subset L_s = W_{\mathcal{O}}/\pi W_{\mathcal{O}} \quad \ast \text{ Let } \sigma_i := L_i/L_{i-1}$$

then $\sigma_i \cong \hat{\otimes}_{v|p} \det^{m_{v,i}}$. $\text{Sym}^{n_{v,i}} \mathbb{F}^2$ for some uniquely determined integers $m_{v,i} \in \{0, \dots, p-2\}$ and $n_{v,i} \in \{0, \dots, p-1\}$.

The patching construction of Kisin-Taylor-Wiles gives us (for some integer $h+j$) ([Kisin]) and formal variables y_1, \dots, y_{h+j}):

- an $(R_{\infty}^i, \mathcal{O}[y_1, \dots, y_{h+j}])$ -bimodule M_{∞} , finite free over $\mathcal{O}[y_1, \dots, y_{h+j}]$.
- filtration of $M_{\infty} / \pi M_{\infty}$: $0 = M_{\infty}^0 \subset M_{\infty}^1 \subset \dots \subset M_{\infty}^s = M_{\infty} / \pi M_{\infty}$ such that $M_{\infty}^i / M_{\infty}^{i-1}$ is a finite free $\mathbb{F}[y_1, \dots, y_{h+j}]$ -module whose isom. class as an $(R_{\infty}^i, \mathbb{F}[y_1, \dots, y_{h+j}])$ -bimodule depends only on the isom. class of σ_i as a $\prod_{v|p} \mathrm{GL}_2(\mathcal{O}_{F_v})$ -module and not on the choices of a_v, b_v, τ_v .

Remark: one can view patching as a covariant exact functor which takes fin. gen. \mathcal{O} -modules with $\prod_{v|p} \mathrm{GL}_2(\mathcal{O}_{F_v})$ -action to coherent sheaves on $\mathrm{Spf} R_{\infty}$ with certain properties. This ~~all~~ appears in [Emerton-Gee-Savitt]

For any sense wt $\sigma_{m,n}$, define $\mu_{m,n}(\bar{v}) := e(R_{cr}^{\square, \Psi}(m, n, \mathbb{1}, \bar{v}) / \pi)$

and $e_{m,n} := Z(R_{cr}^{\square, \Psi}(m, n, \mathbb{1}, \bar{v}) / \pi)$

We can now prove (BM conj. \Rightarrow geometric version):

Fix a, b, τ and $*$ and put $a_v = a, b_v = b, \tau_v = \tau$ for each $v|p$.

We collect two facts from [Kisin] ("The Fontaine-Mazur conj. for GL_2 ")

(1) M_{∞} is a faithful R_{∞}^i -module which has rank 1 at all generic points of R_{∞}^i

(2) $e(R_{\infty}^i / \pi) = e(M_{\infty} / \pi M_{\infty}, R_{\infty}^i / \pi)$

Let $R_{\infty}^{i,i} := \left(\hat{\bigotimes}_{v|p} R_{cr}^{\square, \Psi}(m_{v,i}, n_{v,i}, \mathbb{1}, \bar{v}_{\mathrm{GF}_v}) / \pi \right) [x_1, \dots, x_g]$ (regarded as a quotient of R_{∞})

One then proves that the action of R_{∞}^i on $M_{\infty}^i / M_{\infty}^{i-1}$ factors through $R_{\infty}^{i,i}$.

By a general fact on cycles (product formula) we have

$$\textcircled{a} \quad Z(R_{\infty}^i / \pi) = \left(\prod_{v|p} Z(R_{cr}^{\square, \Psi}(a_v, b_v, \tau_v, \bar{v}) / \pi) \right) \times Z(\mathrm{Spec} \mathbb{F}[x_1, \dots, x_g])$$

We identify $M_{\infty}/\pi M_{\infty}, M_{\infty}^i/M_{\infty}^{i-1}$, etc with corresponding sheaves on $\text{Spec } R_{\infty}/\pi$.

we have: (b)
$$Z(M_{\infty}/\pi M_{\infty}) = \sum_i Z(M_{\infty}^i/M_{\infty}^{i-1}) \geq \sum_i Z(\text{Spec } R_{\infty}^{1,i}/\pi) = \prod_{\nu/p} \sum_{m,n} a_{m,n}^* e_{m,n} \times \sqrt{Z(\text{Spec } F[x_1, \dots, x_g])}$$

by general fact on cycles support of $M_{\infty}^i/M_{\infty}^{i-1}$ coincides with gen. reduced closed subscheme $\text{Spec } R_{\infty}^{1,i}$ of $\text{Spec } R_{\infty}/\pi$ (proved by Gee, Kisin) by product formula for cycles

On the other hand, from (1) of Kisin above ~~and some results~~ (+ some results) one can deduce (c) $Z(R_{\infty}^1/\pi) = Z(M_{\infty}/\pi M_{\infty}), R_{\infty}^1/\pi$

Putting all these together we get:

$$\left(\prod_{\nu/p} Z(R_{\infty}^{1,\psi}(a,b,\tau,\bar{r})/\pi) \right) \times \sqrt{Z(\text{Spec } F[x_1, \dots, x_g])} \stackrel{(a)}{=} Z(R_{\infty}^1/\pi) \stackrel{(c)}{=} Z(M_{\infty}/\pi M_{\infty}) \geq \prod_{\nu/p} \sum_{m,n} a_{m,n}^* e_{m,n} \times \sqrt{Z(\text{Spec } F[x_1, \dots, x_g])}$$

But when we pass to the corresponding multiplicities, then (2) of Kisin and original Breuil-Mézard conjecture imply that we have in fact an equality above. Thus

$$\prod_{\nu/p} Z(R_{\infty}^{1,\psi}(a,b,\tau,\bar{r})/\pi) = \prod_{\nu/p} \sum_{m,n} a_{m,n}^* e_{m,n}$$

and hence $Z(R_{\infty}^{1,\psi}(a,b,\tau,\bar{r})/\pi) = \sum_{m,n} a_{m,n}^* e_{m,n}$ as required \square

§3. The Breuil-Mézard conjecture for GL_n [reminder: ϵ has HT at -1]

Notations: K/\mathbb{Q}_p finite, O_K, k res. field, E/\mathbb{Q}_p finite, $O = O_E, \pi$ unif., F res. field $\Gamma: G_K \rightarrow GL_n(F)$ continuous. Assume E suff. large, in particular contains the images of all embeddings $K \hookrightarrow \bar{\mathbb{Q}_p}$. Let $Z_+^n = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}_+^n \mid \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n\}$. For any $\lambda \in \mathbb{Z}_+^n$, view λ as a dominant character of GL_n/O and let

$M_{\lambda}^1 := \text{Ind}_{B_n}^{GL_n}(\omega_{\lambda}) / O_n$ be the algebraic O_K -rep of GL_n , where

B_n standard Borel subgp, ω_{λ} the lowest elt of the weyl group.

write M_{λ} for M_{λ}^1 evaluated on O_K (rep. of $GL_n(O_K)$). For any $\lambda \in (\mathbb{Z}_+^n)^{\text{Hom}_{\mathbb{Q}_p}(K, E)}$

write $L_{\lambda} = \bigotimes_{\tau: K \hookrightarrow E} M_{\lambda, \tau} \otimes_{O_{K, \tau}} O$ (O -rep. of $GL_n(O_K)$)

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k-rep of $GL_n(k)$

For $a \in \mathbb{Z}_+^n$ with $p-1 \geq a_i - a_{i+1} \forall 1 \leq i \leq n-1$, let $P_a = (\text{Ind}_{B_n}^{GL_n} (w_0 a))$ evaluated on k

$N_a =$ (irreducible) sub- k -rep. of P_a generated by the highest weight vector

N_a will also mean N_a evaluated on k .

We say that $a \in (\mathbb{Z}_+^n)^{\text{Hom}(k, \mathbb{F})}$ is a Serre weight if

- $\forall \sigma \in \text{Hom}(k, \mathbb{F}) \forall 1 \leq i \leq n-1 : p-1 \geq a_{\sigma, i} - a_{\sigma, i+1}$
- $\forall \sigma \in \text{Hom}(k, \mathbb{F}) : 0 \leq a_{\sigma, n} \leq p-1$ and not all $a_{\sigma, n} = p-1$.

For a Serre wt $a \in (\mathbb{Z}_+^n)^{\text{Hom}(k, \mathbb{F})}$ define irreducible \mathbb{F} -rep of $GL_n(k)$

$$F_a := \bigotimes_{\tau \in \text{Hom}(k, \mathbb{F})} N_{a_\tau} \otimes_{k, \tau} \mathbb{F}$$

By [Herzig] ('Serre wts') we have that F_a are absolutely irred., pairwise non-iso. and every irred. \mathbb{F} -rep. of $GL_n(k)$ is of the form F_a for some a .

Call $\lambda \in (\mathbb{Z}_+^n)^{\text{Hom}_{\mathbb{Q}_p}(K, E)}$ a lift of Serre wt $a \in (\mathbb{Z}_+^n)^{\text{Hom}(k, \mathbb{F})}$ if $\forall \sigma \in \text{Hom}(k, \mathbb{F})$

$\exists \tau \in \text{Hom}_{\mathbb{Q}_p}(K, E)$ lifting σ s.t. $\lambda_\tau = a_\sigma$ and \forall for all other $\tau' \in \text{Hom}_{\mathbb{Q}_p}(K, E)$ lifting σ we have $\lambda_{\tau'} = 0$.

We have a partial ordering of Serre wts \leq , where $b \leq a$ if $a-b$ is a sum of positive simple roots.

lemma: a lift of a , then $L_\lambda \otimes_{\mathbb{Q}_p} \mathbb{F}$ has socle F_λ and every Jordan-Hölder factor of $L_\lambda \otimes_{\mathbb{Q}_p} \mathbb{F}$ is of the form F_b with $b < a$.

Let $\tau: I_K \rightarrow GL_n(E)$ rep. with open kernel which extends to \mathbb{W}_K . Take $\lambda \in (\mathbb{Z}_+^n)^{\text{Hom}_{\mathbb{Q}_p}(K, E)}$

Let $\bar{\tau}: G_K \rightarrow GL_n(\mathbb{F})$ be a cont. rep. If E/E' finite, say that a pot. crystalline rep. $\rho: G_K \rightarrow GL_n(E')$ has Hodge type λ if for each $\tau: K \hookrightarrow E$

(set of Hodge wts of ρ)
$$\text{HT}_\tau(\rho) = \{ \lambda_{\tau, 1} + n - 1, \lambda_{\tau, 2} + n - 2, \dots, \lambda_{\tau, n} \}$$

say that ρ has inertial type τ if $\text{WD}(\rho)|_{I_K} \cong \tau$

Proposition: $\forall \lambda, \tau \exists$ unique quotient $R_{\bar{\tau}, \lambda, \tau}^\square$ of the universal lifting \mathcal{O} -algebra $R_{\bar{\tau}}^\square$ for $\bar{\tau}$ with the following properties:

(1) $R_{\overline{F}, \lambda, \tau}^{\square}$ is reduced, p -torsion free, $R_{\overline{F}, \lambda, \tau}^{\square}[\frac{1}{p}]$ is formally smooth and equidimensional of dim $n^2 + [K:Q_p] \approx \frac{n(n-1)}{2}$

(2) E'/E finite, \mathcal{O} -alg. hom. $R_{\overline{F}}^{\square} \rightarrow E'$ factors through $R_{\overline{F}, \lambda, \tau}^{\square}$ if and only if the corresponding rep. $G_K \rightarrow GL_n(E')$ is pot. crystalline of Hodge type λ and inertial type τ

(3) $R_{\overline{F}, \lambda, \tau}^{\square} / \pi$ is equidimensional.

Write $R_{\overline{F}, \lambda}^{\square} = R_{\overline{F}, \lambda, \Pi}^{\square}$. To formulate BM conjecture we need inertial LLC:

Conjecture (inertial LLC) If τ is an inertial type, then there is a finite-dim smooth irred. \overline{Q}_p -rep $\sigma(\tau)$ of $GL_n(\overline{Q}_K)$ s.t. if $\hat{\tau}$ is any Frobenius-semisimple Weil-Deligne rep. of W_K over \overline{Q}_p , then $(\text{res}_p^{-1}(\hat{\tau}) \otimes \det^{(\frac{n-1}{2})})_{|GL_n(\overline{Q}_K)}$ contains $\sigma(\tau)$ as a subrep. if and only if $\hat{\tau}|_{I_K} \cong \tau$ and $N=0$ on $\hat{\tau}$.
If $p > n$ then $\sigma(\tau)$ is unique.

Remark: $n=2$: Henniart; for general n Paskunas proved it for supercuspidal reps. Now, enlarge E if necessary, to have $\sigma(\tau)$ defined over E .

Let L_{τ} be $GL_n(\overline{Q}_K)$ -stable \mathcal{O} -lattice in $\sigma(\tau)$. Set $L_{\lambda, \tau} := L_{\tau} \otimes_{\mathcal{O}} M_{\lambda}$ (finite free \mathcal{O} -mod. with $GL_n(\overline{Q}_K)$ -action). Write

$$(L_{\lambda, \tau} \otimes_{\mathcal{O}} \mathbb{F})^{\text{ss}} \cong \bigoplus_{\mathfrak{a}} \mathbb{F}_{\mathfrak{a}}^{n_{\mathfrak{a}}} \quad \text{where } \mathfrak{a} \text{ runs over some wts in } (\mathbb{Z}/p\mathbb{Z})^{\text{Hom}(L, \mathbb{F})} \text{ and } n_{\mathfrak{a}} \in \mathbb{Z}_{\geq 0}$$

Then the Breuil-Mézard conj. is:

Conjecture: There exist integers $\mu_{\mathfrak{a}}(\overline{r})$ depending only on \overline{r} and \mathfrak{a} s.t.

$$e(R_{\overline{F}, \lambda, \tau}^{\square} / \pi) = \sum_{\mathfrak{a}} n_{\mathfrak{a}} \mu_{\mathfrak{a}}(\overline{r})$$

Geometric version of the conjecture: For each \mathfrak{a} there is a cycle $\mathcal{C}_{\mathfrak{a}}$ depending only on \overline{r} and \mathfrak{a} s.t. $Z(R_{\overline{F}, \lambda, \tau}^{\square} / \pi) = \sum_{\mathfrak{a}} n_{\mathfrak{a}} \mathcal{C}_{\mathfrak{a}}$.

Remark 1: One can compute $\mu_{\mathfrak{a}}(\overline{r})$ and $\mathcal{C}_{\mathfrak{a}}$ inductively (assuming the conjecture)

Remark 2: one can formulate also both conjectures for rings with param. reps with fixed determinant. Both are equivalent (at least for $p > n$) because if $p > n$ then $R_{\overline{F}, \lambda, \tau}^{\square} \cong R_{\overline{F}, \lambda, \tau}^{\square}[\chi]$ (χ is well-chosen)

Remark 3: Assuming "inertial LLC" one can again prove (BM conj. \Rightarrow geom. version) by patching. proof of this is in [Emerton-Gee] lemma 4.3.1. \square