

We recall some notations first:

- F CM with  $F^+$  tot. real subfield s.t.  $F/F^+$  unramified at all finite places; Every  $v/p$  of  $F^+$  splits in  $F$ ;  $[F^+:\mathbb{Q}]$  is even.

Let  $G/F^+$  be an outer form of  $GL_2$ ,  $G_F \cong GL_2/F$ , which is  $U(2)$  at infinite places (defined as in previous talk). Fix a model of  $G$  over  $\mathcal{O}_{F^+}$ .

Fix abs. irred.  $\bar{\tau}: G_F \rightarrow GL_2(F)$  s.t.  $\bar{\tau}^c \cong \bar{\tau}^v \bar{\epsilon}^{-1}$  where

(universal Hecke algebra)

$$\bar{\tau}^c(g) = \bar{\tau}(cg), \quad c \in \text{Gal}(F/F^+) \text{ generator}$$

$\bar{\epsilon}$  mod  $p$  cyclotomic character.

We assume that  $\bar{\tau}$  is automorphic, i.e. there is a maximal ideal  $m$  of  $\mathbb{T}_{\lambda, T, \text{univ}}$  s.t.  $S_{\lambda, T}(\mathbb{U}, \theta)_m \neq 0$  for some  $(\lambda, T)$  and  $\tau(\text{Frob}_w)$  has char. polynomial equal to  $X^2 - T_w^{(1)}X + (Nw)T_w^{(2)}$  in  $\mathbb{T}_{\lambda, T, \text{univ}}/m[[X]]$  for all  $v \notin T$  s.t.  $v = w w^c$  in  $F$

$\mathbb{T}_F$   
places of  $F^+$ .

Here  $T$  is a finite set of places of  $F^+$  which split in  $F$ , considering places dividing  $p$  and all the places at  $v$  which split in  $F$  for which  $\mathbb{U}_v \neq G(\mathcal{O}_{F_v^+})$ .  $\mathbb{U}$  is a good compact open subgroup of  $G(\mathbb{A}_{F^+}^\infty)$ , which we fix throughout. There are some technical conditions on  $\mathbb{U}$  assumed which we do not mention. They make  $S_{\lambda, T}(\mathbb{U}, \theta)_m \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$  locally free over  $\mathbb{T}_{\lambda, T}^T(\mathbb{U}, \theta)_m[\frac{1}{p}]$ . (Recall  $\mathbb{T}_{\lambda, T}^T(\mathbb{U}, \theta) = \text{Im of } \mathbb{T}_{\lambda, T, \text{univ}} \text{ in } (\text{End } S_{\lambda, T}(\mathbb{U}, \theta))_{\mathcal{O}_{F_v^+}}$ )

As before we take  $G_2$  to be  $GL_2 \times GL_1 \times \{1, j\}$  with action of  $\{1, j\}$  by  $j(g, \mu) j^{-1} = (\mu^{(t_j g)^{-1}}, \mu)$ . Put  $\nu: G_2 \rightarrow GL_1: (g, \mu) \mapsto \mu$ . Extend  $\bar{\tau}$  to  $\bar{\rho}: G_{F^+} \rightarrow G(F)$  with  $\nu \circ \bar{\rho} = \bar{\epsilon}^{-1}$ ,  $j \mapsto -1$ .   
 $\bar{\rho}|_{G_F} = (\bar{\tau}, \bar{\epsilon}^{-1})$

Choose  $E$  suff. large finite ext. of  $\mathcal{O}_{F^+}$ , with  $\mathcal{O} = \mathcal{O}_E$ , res. field  $F$ .

Recall that our goal is to describe Hilbert-Samuel multiplicity of certain deformations rings associated to  $(\bar{r}, \lambda, T)$ .

This talk: define for each Serre weight  $a \in (\mathbb{Z}_+^{*2})^{\text{Hom}(\mathbf{h}, F)}$

a non-negative integers  $\mu_a(\bar{r})$  which will appear in the RHS of the Breuil-Mézard conjecture. We do it by patching of KTW.

Deformations:  $\mathcal{C}_0 = \text{cat. of local complete Noetherian } O\text{-algebras with res. field isom. to } F$ .  $S = \text{set of places of } F^+ \text{ which split in } F \text{ containing all the places dividing } p$ . We regard  $\bar{g}$  as a rep. of  $G_{F^+, S} = \text{Gal}(F(S)/F)$ , where  $F(S)$  is the max ext. of  $F$  unramified outside places over  $S$ .

- Lifting of  $\bar{g}$  to an object  $A \in \mathcal{C}_0 = \text{cont. homom. } g: G_{F^+, S} \rightarrow GL_2(A) \text{ lifting } \bar{g} \text{ and s.t. } \gamma \circ g = \epsilon^{-1} \text{ ( } \epsilon \text{ - } p\text{-adic cyclo char.)}$
- $\varrho_1, \varrho_2 \circ \text{liffts of } \bar{g} \text{ are equivalent if they are conjugate by an element of } \ker(GL_2(A) \rightarrow GL_2(F))$
- deformation of  $\bar{g}$  = equivalence class of liftings.

If  $T \subset S$ , then:

- $T$ -framed lifting =  $(\varrho, \{\alpha_v\}_{v \in T})$  where  $\varrho$  lifting of  $\bar{g}$  and  $\alpha_v \in \ker(GL_2(A) \rightarrow GL_2(F))$  for  $v \in T$ .
- $(\varrho_1, \{\alpha_{1,v}\}_{v \in T}) \sim (\varrho_2, \{\alpha_{2,v}\}_{v \in T})$  equivalent if  $\exists \beta \in \ker(GL_2(A) \rightarrow GL_2(F))$  with  $\varrho_2 = \beta \varrho_1 \beta^{-1}$  and  $\alpha_{2,v} = \beta \alpha_{1,v} \beta^{-1}$  for  $v \in T$ .
- $T$ -framed deformation = equivalence class of  $T$ -framed liftings.

For each  $v \in T$ , choose  $\tilde{v}$  place of  $F$  above  $v$ .  $\tilde{T} = \{\tilde{v}\}_{v \in T}$   
 $\forall_{v \in T} R_{\tilde{v}}^D = \text{universal } O\text{-lifting ring of } \tilde{T} / G_{F^+}$ . For each  $v \in S_p = \{v/p\}_{v \in T}$   
let  $R_{\tilde{v}}^{D, \alpha_v, \tau_v} := R_{\tilde{v} / G_{F^+}}^{D, \alpha_v, \tau_v}$ .

P. Choochit ~~Kisin-Taylor-Wiles~~ patching (2)

We have a deformation problem:  $([CHT]) \quad \{R_{S,T}\}_{v \in R, v \in S_p} = T$

$$S = (F/F^+, T, \tilde{T}, \partial, \bar{\rho}, \epsilon^{-1}, \{R_v^\square\}_{v \in R} \cup \{R_v^0, \lambda_v, \tau_v\}_{v \in S_p})$$

which gives a universal deformation  $\mathcal{G}_S^{\text{univ}}: G_{F^+, T} \rightarrow \mathcal{G}_2(R_S^{\text{univ}})$   
 which is universal for nipp deformations  $\tilde{\rho}$  of  $\bar{\rho}$  with  $\nu \circ \tilde{\rho} = \epsilon^{-1}$  and  
 s.t.  $V_{v \in T}$  the point of  $\text{Spec } R_v^\square$  corr. to  $r|_{G_{F_v}}$  is a point  
 of  $\text{Spec } R_{S,T}$  (in general,  $R_{S,T}$  is a quotient of  $R_v^\square$ ).

By lifting of previous talk and universality of  $\mathcal{G}_S^{\text{univ}}$  we  
 get a (surjective!) map  $R_S^{\text{univ}} \xrightarrow{\text{lift}} \mathbb{T}_{A,T}^T(\mathcal{U}, \partial)_m$ .

Here  $T = R \amalg S_p$ .

For patching we will need some auxiliary set of primes. To find them  
 one needs to put additional condition on  $\mathbb{T}$  ( $\mathbb{T}(G_{F(\tilde{\chi}_p)})$  adequate).

Let  $(Q, \tilde{Q}, \{V_v\}_{v \in Q})$ :  
 •  $Q$  finite set of places of  $F^+$  disjoint from  $T$   
 and consists of places which split in  $F$   
 •  $\tilde{Q} = \{v \in Q\}$  choice of a single place of  $F$   
 above each  $v \in Q$

$\forall_{v \in Q} \quad \tilde{r}|_{G_{F_v}} \cong \overline{\chi_v} \oplus \overline{\chi'_v}$  where  $\overline{\chi_v} \neq \overline{\chi'_v}$  and  $Nv \equiv 1 \pmod{p}$

For each  $v \in Q$  let  $R_v^{\square, \overline{\chi_v}}$  = quotient of  $R_v^\square$  corr. to lifts  $v|_{G_{F_v}} \rightarrow GL_2(A)$   
 which are  $\ker(GL_2(A) \rightarrow GL_2(F))$ -conjugate to a lift of the form  $\gamma \oplus \gamma'$   
 where  $\gamma$  lifts  $\chi_v$  and  $\gamma'$  is an unramified lift of  $\overline{\chi'_v}$ . We have

a deformation problem:  $S_Q = (F/F^+, T \cup Q, \tilde{T} \cup \tilde{Q}, \partial, \bar{\rho}, \epsilon^{-1}, \{R_v^\square\}_{v \in R} \cup \{R_v^0, \lambda_v, \tau_v\}_{v \in S_p} \cup \{R_v^{\square, \overline{\chi_v}}\}_{v \in Q})$

Let  $R_{S,Q}^{\text{univ}}$  corresponding univ. deform. nipp

$R_{S,Q}^{\square, T}$  corr. univ.  $T$ -framed deform. nipp

Define  $R^{\text{loc}} := (\bigotimes_{v \in S_p} R_v^{\square, \lambda_v, \tau_v}) \hat{\otimes} (\bigotimes_{v \in R} R_v^{\square, \lambda_v, \tau_v})$ ,  $\hat{\otimes}$  is over  $\mathcal{O}$ .  
 One can prove that  $R^{\text{loc}}$  is equidimensional and  $R^{\text{loc}}[\frac{1}{p}]$  is formally smooth.

For each finite place  $w$  of  $F$ , let  $U_0(w) = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod w \right\} \subseteq GL_2(O_{F,w})$

$$U_1(w) = \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod w \right\} \subseteq GL_2(O_{F,w})$$

$U_0(Q) = \prod_v U_0(Q)_v$  and  $U_1(Q) = \prod_v U_1(Q)_v$  comp. open subgp of  $GL_2(\mathbb{A}_{F,+}^{\text{tor}})$  defined by ( $i=0,1$ )  $U_i(Q)_v = U_v$  if  $v \notin Q$ ,  $U_i(Q) = U_i(\tilde{v})$  if  $v \in Q$

We have:  $\prod_{\lambda, T}^{T \times Q} (U_1(Q), 0) \rightarrow \prod_{\lambda, T}^{T \times Q} (U_0(Q), 0) \xrightarrow{\text{actually a pullback by } G(O_{\lambda}) \cong GL_2(O_{\lambda})} \prod_{\lambda, T}^{T \times Q} (U, 0) \hookrightarrow \prod_{\lambda, T}^T (U, 0)$

Let  $m'_Q$  be the ideal in  $\mathfrak{J}$  corr. to  $m$  in  $\mathfrak{J}$ .  
One can enlarge  $\prod_{\lambda, T}^{T \times Q} (U_1(Q), 0)$  by ~~certain Hecke operators~~ <sup>(additives)</sup> and get an ideal  $m_Q'$  there gen. by  $m'_Q$  and " $V_{w, \lambda} - A_{\lambda}$ ". (operator)

Let  $\Delta_Q = \max$  p-power order quotient of  $U_0(Q)/U_1(Q)$ .

$R_Q$  = kernel of augmentation map  $O[\Delta_Q] \rightarrow O$ .

One has  $R_{S_Q}^{\text{univ}} / \Delta_Q \cong R_S^{\text{univ}}$ ,  $R_{S_Q}^{\text{DT}} / \Delta_Q \cong R_S^{\text{DT}}$

and  $S_{\lambda, T}(U_1(Q), 0)_{m_Q'}$  is free over  $O[\Delta_Q]$ .

with  $S_{\lambda, T}(U_1(Q), 0)_{m_Q'} / \Delta_Q \cong S_{\lambda, T}(U, 0)_m$ .

Patching:  $\overline{r}(G_{F(\xi_p)})$  adequate gives an existence of integer  $q \geq [F^+ : \mathbb{Q}]$  and  $\#_{N \geq 1}$  a tuple  $(Q_N, \tilde{Q}_N, \{V_v\}_{v \in Q_N})$  s.t. as above s.t.

- $\#Q_N = q \#_N$  ; •  $N_v \equiv 1 \pmod{p^N} \forall v \in Q_N$
- $R_{S_{Q_N}}^{\text{DT}}$  can be topologically generated over  $R^{\text{loc}}$  by  $q[F^+ : \mathbb{Q}]$  elts.

Let  $W = O[[X_{v,i,j} : v \in T, i, j = 1, 2]]$ .

$$M = S_{\lambda, T}(U, 0)_m \otimes_{R_S^{\text{univ}}} R_S^{\text{DT}}$$

$$M_N = S_{\lambda, T}(U_1(Q), 0)_{m_{Q_N}} \otimes_{R_{S_{Q_N}}^{\text{univ}}} R_{S_{Q_N}}^{\text{DT}}$$

Then  $M_N$  is a finite free  $W[\Delta_{Q_N}]$ -module with  $M_N / \Delta_{Q_N} \cong M$ .

compatible with  $R_{S_{Q_N}}^{\text{DT}} / \Delta_{Q_N} \cong R_S^{\text{DT}}$

Recall that for  $(\lambda, T)$  we have defined a lattice  $L_{\lambda, T}$ .

Fix a filtration by  $\mathbb{F}$ -subspaces

$$0 = L_0 \subset L_1 \subset \dots \subset L_s = L_{\lambda, \mathbb{Z}} \otimes \mathbb{F}$$

s.t.  $L_i$  is  $G(\mathcal{O}_{F^+})$ -stable and  $H_{i=0, \dots, s} \cap \sigma_i = L_{i, \mathbb{Z}} / L_i$  is an absolutely irreduc. rep. of  $G(\mathcal{O}_{F^+})$ .

This induces filtration on  $S_{\lambda, T}(\mathfrak{l}, \theta)_m \otimes \mathbb{F}$  and hence on

$$M \otimes_{\Theta} \mathbb{F}: \quad 0 = M^0 \subset M^1 \subset \dots \subset M^s = M \otimes \mathbb{F}$$

$$M_N \otimes_{\Theta} \mathbb{F}: \quad 0 = M_N^0 \subset M_N^1 \subset \dots \subset M_N^s = M_N \otimes \mathbb{F}$$

Let  $g = q - [F^+ : \mathbb{Q}]$  and

$$\Delta_\infty = \mathbb{Z}_p^q$$

$$R_\infty = R^{\text{loc}}[x_1, \dots, x_g]$$

$$R_\infty^i = (\bigotimes_{v \in T} R_v^0)[x_1, \dots, x_g]$$

$$S_\infty = W[\Delta_\infty] \quad (\text{formally smooth over } \Theta)$$

and let  $S_\infty \rightarrow \Theta: x_{v, i, j} \mapsto 0, \quad \sigma \in \Delta_\infty \mapsto 1$ .  
 let  $\mathbb{I} = \ker(S_\infty \rightarrow \Theta)$ .

Now patching of Kisin-Taylor-Wiles gives:

- an  $S_\infty$ -module  $M_\infty$  which is simultaneously  $R_\infty$ -module s.t. the image of  $R_\infty$  in  $M_\infty$  is an  $S_\infty$ -algebra.
- filtration by  $R_\infty$ -modules:  $0 = M_\infty^0 \subset M_\infty^1 \subset \dots \subset M_\infty^s = M_\infty \otimes \mathbb{F}$   
 whose graded pieces are finite free  $S_\infty$ -modules.
- surjection of  $R^{\text{loc}}$ -algebras  $R_\infty/\mathbb{I}R_\infty \rightarrow R_S^{\text{ur}}$
- isom. of  $R_\infty$ -modules  $M_\infty/\mathbb{I}M_\infty \xrightarrow{\sim} M$  which identifies  $M^i$  with  $M_\infty^i/\mathbb{I}M_\infty^i$ .

When  $\alpha$  is a global semi weight and  $L_i/L_{i-1} \cong F_\alpha$  we denote the  $R_\infty/\mathbb{I}R_\infty$  module  $M_\infty^i$  by  $M_\infty^\alpha$ .

We set  $M_\alpha(\bar{F}) = 2^{-[F^+]} e_{R_\infty/\mathbb{I}R_\infty}(M_\infty^\alpha)$ .

Now one can prove that  $\mu_a^!(\bar{F})$  is a non-negative integer.

~~Moreover~~ let  $\varrho_1, \varrho_2 : G_K \rightarrow GL_2(\mathbb{Q})$  ( $K/\mathbb{Q}_p$  finite). We say that  $\varrho_1$  connects to  $\varrho_2$  if  $\varrho_1, \varrho_2$  are both crystalline of the same Hodge type  $\lambda$ ,

- $\bar{\varrho}_1 \cong \bar{\varrho}_2$  and  $\varrho_1$  and  $\varrho_2$  define points on the same irreducible component of  $R_{\varrho_1}^{D, \lambda} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p}$ .

We say that  $\varrho : G_K \rightarrow GL_2(\mathbb{Q})$  is diagonal if it is a direct sum of crystalline characters.  $\varrho$  is diagonalizable if it connects to some diagonal rep. Finally,  $\varrho$  is potentially diagonalizable if  $\mathbb{Z}/h/K$  finite s.t.  $\varrho|_{G_L}$  is diagonalizable.  $\varrho : G_K \rightarrow GL_2(\mathbb{Z})$  is pot. diag. if some  $G_K$ -lattice of  $\varrho$  is (condition independent of the choice of lattice).

Now Gee-Kisin proves

Prop: Suppose  $\bar{F} : G_F \rightarrow GL_2(\mathbb{F})$  is as before and moreover for each place w/p, every lift of  $\bar{F}|_{G_{F_v}}$  of Hodge type  $\lambda_{\bar{F}}$  and Galois type  $\tau_v$ , is potentially diagonalizable. Then

$$(*) \quad e(R_{\varrho_0}/\pi R_{\varrho_0}) = \sum_{i=1}^s \mu_{\alpha_i}^!(\bar{F})$$

where  $\alpha_i$  is a global semi-weight with  $L_i/L_{i-1} \cong F_{\alpha_i}$ .

On the proof: there are 2 ingredients: one is the lemma which proves equivalence of  $(*)$  with some other conditions on  $M, M_{\varrho_0}$ . Then one uses a result on construction of automorphic lifts of  $\bar{F}$  to conclude (here is needed potential diagonalizability).