

# Cherednik GDT "On patching functors"

## S1. Introduction

Notations:  $p$  prime,  $K/\mathbb{Q}_p$  finite,  $I_K \subset G_K = \text{Gal}(K/K)$  inertia subgp.

$F_v$  finite unramified ext. of  $\mathbb{Q}_p$  (usually  $F_v = \text{completion of } e$  number field  $F$ )

$k_v = \text{res. field}$ ,  $O_{F_v}$  integers ( $O_{F_v} = W(k_v)$ ). Write  $f = [F_v : \mathbb{Q}_p]$ ,  $q_v = p^f$ ,  $e = p^{f-1}$ .

Coefficient rings:  $E/\mathbb{Q}$  finite,  $\mathcal{O} = O_E$ ,  $F = \mathcal{O}/\bar{\omega}_E$ ,  $\bar{\omega}_E$  uniformized.

We define the fundamental character  $c_F$  of  $I_{F_v}$  of level  $f$  as

$$I_{F_v} \xrightarrow{\text{Art}_{F_v}} \mathcal{O}_{F_v}^\times \rightarrow k_v^\times \xrightarrow{\bar{\kappa}_0} F^\times$$

similarly define  $\omega_{2f}$  for  $\bar{\kappa}_0' : k_v^\times \hookrightarrow F$  and  $L_v/F_v$  qud. unram. ext. with res. field  $l_v$ . ( $l_v$ )

Definition: We say that a continuous rep  $\bar{\rho} : G_{F_v} \rightarrow GL_2(F)$  is generic if  $\bar{\rho}|_{I_{F_v}}$  is isomorphic up to a twist to a rep. of one of the following forms:

$$(i) \begin{pmatrix} \omega_p^{\sum_{j=0}^{f-1} (r_j+1)p^j} & * \\ 0 & 1 \end{pmatrix} \text{ with } 0 \leq r_j \leq p-3 \text{ if } r_j \text{ are not all equal to } 0, \text{ and not all equal to } p-3.$$

$$(ii) \begin{pmatrix} \omega_{2f}^{\sum_{j=0}^{f-1} (r_j+1)p^j} & 0 \\ 0 & \omega_{2f}^{q_v \sum_{j=0}^{f-1} (r_j+1)p^j} \end{pmatrix} \quad \begin{matrix} q_v = p^f \\ 1 \leq r_0 \leq p-2, \quad 0 \leq r_j \leq p-3 \quad \forall j > 0 \end{matrix}$$

Remark: genericity is used to have regular some weights (work of T. Gee)

Serre weight: irreducible  $F$ -rep. of  $GL_2(k_v)$ :

$$\overline{\sigma}_{t,s} = \bigotimes_{j=0}^{f-1} \det^{-t_j} \text{Sym}^{s_j} k_v^\times \otimes_{k_v, \bar{k}_v} F. \quad \begin{bmatrix} \bar{k}_v \\ \text{some embedding} \end{bmatrix}$$

where  $0 \leq r_j, t_j \leq p-1$  and not all  $t_j$  are equal to  $p-1$ .

Definition:  $\overline{\sigma}_{t,s}$  is regular if no  $s_j = p-1$ .

Recall that one may associate to  $\bar{\rho}$  a set of some sets  $D(\bar{\rho})$  for which I conjecturally (conjecture of Buzzard - Diamond - Jarvis)  $\bar{\rho}$  is modular/etale-morphic. (in the global setting)

$$\text{when } \bar{\rho} = \bar{\sigma}|_{G_F} \text{ for some global } \bar{\sigma}$$

Goal of this talk: show BDJ conj. for generic reps. in different context by using patching functors from [EGS].

we recall also: (Gee - Liu - Savitt)  $\bar{\sigma} \in D(\bar{\rho})$  iff  $\bar{\rho}$  has a crystalline lift of Hodge type  $\bar{\sigma}$  (i.e.  $\bar{\sigma} = \bar{\sigma}_{\pm, \pm}$  as HT type  $(\pm, \pm, \pm + 1)$ )

Recall that  $(L/\mathbb{Q}_\ell, \ell \text{ any prime})$  an inertial type for  $L$  is a two-dim E-rep.  $\tau$  of  $I_L$  with open kernel which extends to  $G_L$ .

Inertial LLC associates to it finite-dim irreducible E-rep.  $\sigma(\tau)$  of  $GL_2(\mathbb{Q}_\ell)$  (normalisation as in Gee - Kisin).

Also, one can do the same for  $\ell \neq p$  and a non-split quasirepresentation  $\bar{\sigma}$  with centre  $L$   $\leadsto$   $\sigma(\tau)$  rep. of  $\mathcal{O}_K^\times$  fin.-dim irreducible over  $E$

A  tame type for us will be an <sup>irred.</sup> E-rep. of  $GL_2(\mathbb{Q}_F) = K$  which arises by inflation from the irred. rep. of  $GL_2(k_v)$ , and moreover it is either principal series or cuspidal type (in general, it can be also one-dim. or a twist of Steinberg). - classification of Diamond

We call inertial types  $\tau$  for which  $\sigma(\tau)$  is ~~not~~ a tame type

~~or~~ a tame inertial type.

For further use, we note a result of Gee:

Proposition 1: Suppose  $\bar{\rho}$  is generic and  $\bar{\sigma} \in D(\bar{\rho})$ . Then there is a non-scalar tame inertial type  $\tau$  s.t.  $JH(\sigma(\tau)) \supseteq D(\bar{\rho})$

+  
Jordan-Hölder decomposition  
of  $\sigma(\tau)$ .

# (2.)

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## §2. Abstract patching functors

Fix prime  $p$ ,  $E/\mathbb{Q}_p$  finite,  $\mathcal{O} = \mathcal{O}_E$ ,  $\mathbb{F}$  ws. field. Local fields are here finite extensions of  $\mathbb{Q}_p$  with  $\ell$  not necessarily  $= p$ .

- $\mathfrak{l}_i$  local fields,  $\mathfrak{l}_i$  res. char. of  $\mathfrak{l}_i$ ,  $\mathcal{O}_{\mathfrak{l}_i}$ .  
if  ~~$\mathfrak{l}_i = p$~~  assume  $\mathfrak{l}_i/\mathbb{Q}_p$  unramified,  $\bar{\varrho}_i : G_{\mathfrak{l}_i} \rightarrow GL_2(\mathbb{F})$
- $V_i$ :  $K$  either compact open of  $GL_2(\mathcal{O}_{\mathfrak{l}_i})$  or  $\mathcal{O}_D^\times$  where  $D_i$  qud. dp with centre  $\mathfrak{l}_i$ .  
if  $\mathfrak{l}_i = p$  we only allow  $GL_2(\mathcal{O}_{\mathfrak{l}_i})$ .  
write  $K = \prod K_i$ ,  $Z_K = Z(K)$ , reduced norm gives  $\begin{matrix} \text{norm.} \\ (\text{determinant}) \end{matrix} Z_K \rightarrow \prod \mathcal{O}_{\mathfrak{l}_i}^\times$ .
- $V_i$ :  $R_i^D$  universal framed def. ring of  $\bar{\varrho}_i$  over  $\mathcal{O}$ .  
 $X_i = \text{Spf } R_i^D$ ,  $R = \bigotimes_{\mathcal{O}} R_i^D$ ,  $X = \text{Spf } R$ . Fix some  $b > 0$   
 $R_\infty = R[[x_1, \dots, x_b]]$ ,  $X_\infty = \text{Spf } R_\infty$ ,  $x_i$  are formal variables.
- $\mathcal{C} = \text{cat. of f.flat. gen. } \mathcal{O}\text{-modules with a const. action of } K$ ,  
 $\mathcal{C}'$  Serre subcat. of  $\mathcal{C}$  (for flexibility, mostly  $\mathcal{C}' = \mathcal{C}$ )
- $V_i$ :  $T_i$  inertial type for  $I_{\mathfrak{l}_i}$ ,  $\sigma^\circ(T_i)$   $\mathcal{O}$ -lattice in  $\sigma(T_i)$ .  
Write  $\sigma^\circ(t) := \bigotimes \sigma^\circ(t_i)$  and suppose  $\sigma^\circ(t) \in \text{ker } \sigma_D^\circ(t_i)$
- If  $\mathfrak{l}_i \neq p$ ,  $R_i^{D,T} = \text{reduced, } p\text{-torsion free quotient of } R_i^D$  corresponding  
to deformations of inertial type  $T_i$ .  
if  $\mathfrak{l}_i = p$ ,  $R_i^{D,T} = \text{reduced, } p\text{-torsion free quotient of } R_i^D$  corr. to potentially  
crystalline deformations of inertial type  $T_i$  and Hodge type  $D$ .  
Write  $R^T = \bigotimes_{\mathcal{O}} R_i^{D,T}$ ,  $R_\infty^T = R^T[[x_1, \dots, x_b]]$ ,  $X_\infty(T) = \text{Spf } R_\infty^T$ ,  $X(T) = \text{Spf } R^T$ .
- Suppose that  $\bar{\sigma} \in \mathcal{C}'$  is killed by  $\bar{\omega}_E$  and  $V_i$  with  $\mathfrak{l}_i = p$ ,  $\bar{\sigma}|_{K_i}$  is  
a direct sum of copies of an irreduc. rep.  $\bar{\sigma}_i$  of  $K_i$ . In this case,  
 $V_i : \mathfrak{l}_i = p$  let  $R_i^{D,\bar{\sigma}} = \text{reduced } p\text{-torsion free quotient of } R_i^D$  corr. to  
crystalline ~~rep~~ deformations of Hodge type  $\bar{\sigma}_i$  and  $V_i : \mathfrak{l}_i = p$   $R_i^{D,\bar{\sigma}} = R_i^D$ .  
Write  ~~$R^{\bar{\sigma}}$~~   $R^{\bar{\sigma}} := \bigotimes_{\mathcal{O}} R_i^{D,\bar{\sigma}}$ ,  $R_\infty^{\bar{\sigma}} = R^{\bar{\sigma}}[[x_1, \dots, x_b]]$ ,  $X_\infty(\bar{\sigma}) = \text{Spf } R_\infty^{\bar{\sigma}}$ .  
Write  $\overline{X_\infty(\bar{\sigma})}$  for the special fibre of  $X_\infty(\bar{\sigma})$ .

Definition: A patching functor is a covariant functor  $M_{\sigma}$  from  $\mathcal{C}$  to the category of coh. sheaves on  $X_0$  s.t.  $\mathcal{V}_{\sigma^0(\tau)}$ ,  $\sigma$  as above :

- $M_{\sigma}(\sigma^0(\tau))$  is p-torsion free and supported on  $X_0(\tau)$  and is maximal Cohen-Macaulay over  $X_0(\tau)$ .
- $M_{\sigma}(\bar{\sigma})$  is supported on  $\overline{X_0(\bar{\sigma})}$  and is max. Cohen-Macaulay  $\overline{X_0(\bar{\sigma})}$ .

Remark: patching functors arise from Kisin-Taylor-Wiles method, but there are others too! (work of Gee et al. in progress).

We need a version of the above notion with a fixed determinant  $\det$ :

- $\mathcal{C}_2 = \text{subset of } \mathcal{C} \text{ of reps. with central char. so that the action of } \mathbb{Z}_K \text{ factors through the natural map to } \prod_i \mathbb{Q}_L^\times \text{ and whose central character lifts the character } \prod_i (\bar{\epsilon} \cdot \det \bar{\rho}_i : I_{L_i}) \circ \text{Art}_{L_i}.$  Let  $\mathcal{C}'_2$  be a semi-subcategory of  $\mathcal{C}_2$ .
- Take some  $\sigma \in \mathcal{C}'_2$ . Fix char.  $\gamma_{\sigma,i} : G_{L_i} \rightarrow \mathcal{O}^\times$  s.t.  $\gamma_{\sigma,i}(Frob_i) = \epsilon_i$  ( $Frob_i = \text{geom. Frob. } \in G_{L_i}$ ),  $d_i \in \mathcal{O}_{L_i}^\times$  fixed elt. lifting  $\det \bar{\rho}_i(Frob_i)$ ) and  $\gamma_{\sigma,i} / I_{L_i} \circ \text{Art}_{L_i}$  = central character of  $\sigma$ . (if  $\sigma$  is not p-power torsion, then  $\gamma_{\sigma,i}$  is uniquely determined).
- $R_i^{D, \gamma_{\sigma}} = \text{quotient of } R_i^D \text{ corr. to liftings with determinant } \gamma_{\sigma,i} \epsilon^{-1}$  and  $X_i^{D, \gamma_{\sigma}} = \text{Spf } R_i^{D, \gamma_{\sigma}}$ .  $R^{D, \gamma_{\sigma}} = \bigotimes_i R_i^{D, \gamma_{\sigma}}$ ,  $X^{D, \gamma_{\sigma}} = \text{Spf } R^{D, \gamma_{\sigma}}$ . Fix some  $h \geq 0$ ,  $R_{\infty}^{D, \gamma_{\sigma}} = R^{D, \gamma_{\sigma}}[[x_1, \dots, x_h]]$ ,  $X_{\infty}^{D, \gamma_{\sigma}} = \text{Spf } R_{\infty}^{D, \gamma_{\sigma}}$ .
- assume that  $\tau_i$  as above satisfy:  $\det \tau_i$  lift  $\bar{\epsilon} \cdot \det \bar{\rho}_i : I_{L_i}$  so that  $\sigma^0(\tau) \in \mathcal{C}_2$  (assume  $\sigma^0(\tau) \in \mathcal{C}'_2$ ).
- define  $R_i^{D, \gamma, \tau}$  quotient of  $R_i^{D, \gamma}$  and  $R^{\gamma, \tau} = \bigotimes_i R_i^{D, \gamma, \tau}$ ,  $R_{\infty}^{\gamma, \tau} = R^{\gamma, \tau}[[x_1, \dots, x_h]]$ ,  $X_{\infty}^{\gamma}(\tau) = \text{Spf } (R_{\infty}^{\gamma, \tau})$ . If  $\bar{\sigma} \in \mathcal{C}'_2$  then we have quotient  $R_i^{D, \gamma, \bar{\sigma}}$  of  $R_i^{D, \bar{\sigma}}$  and  $R^{\gamma, \bar{\sigma}} = \bigotimes_i R_i^{D, \gamma, \bar{\sigma}}$ ,  $R_{\infty}^{\gamma, \bar{\sigma}} = R^{\gamma, \bar{\sigma}}[[x_1, \dots, x_h]]$ ,  $X_{\infty}^{\gamma}(\bar{\sigma}) = \text{Spf } (R_{\infty}^{\gamma, \bar{\sigma}})$ .
- $\overline{X^{\gamma}(\bar{\sigma})}$  = special fibre of  $X^{\gamma}(\bar{\sigma})$ .

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Definition: A fixed determinant patching functor is a covariant exact functor  $M_\infty$  from  $\mathcal{C}_\infty^1$  to cat. of coh. sheaves on  $X_\infty$  s.t.:

- (i)  $H \circ e_{\mathcal{C}_\infty^1}$ ,  $M_\infty(\sigma)$  is supported on  $X_\infty^\sigma$ .
- (ii)  $M_\infty(\sigma^\circ(\tau))$  is p-torsion free and supported in  $X_\infty^\sigma(\tau)$  and is max. Cohen-Macaulay over  $X_\infty^\sigma(\tau)$ .
- (iii)  $M_\infty(\bar{\sigma})$  is supported on  $\overline{X_\infty^\sigma(\bar{\sigma})}$  and is max. coh.-Mac. over  $\overline{X_\infty^\sigma(\bar{\sigma})}$ .

## §3 Generic Buzzard-Diamond-Jarvis conjecture

$\bar{g}: G_{F_v} \rightarrow GL_2(F)$ ,  $\tau$  tame inertial type,  $X_\infty(\tau) = \text{Spf } R^\tau$ .

perem. lifts of  $\bar{g}$  pt. BT of type  $\tau$  and Hodge type 0.

Fix  $\gamma: G_{F_v} \rightarrow O^\times$  lifting  $\bar{\epsilon} \det g$ ,  $\gamma|_{I_{F_v}} = \det \tau$ ,  $X^\gamma(\tau) = \text{Spf } R^{\gamma, \tau}$

(lifts with determinant  $\gamma \in \gamma^{-1}$ ). Then since at  $\bar{\sigma}$  we have also  $X(\bar{\sigma})$ ,  $X^\gamma(\bar{\sigma})$ : crystalline lifts of Hodge type  $\bar{\sigma}$ .

Theorem 1:  $\bar{g}$  generic,  $\tau$  tame inertial type. Then mod  $\bar{\omega}_E$  fibre of  $X(\tau)$  is the union of the mod  $\bar{\omega}_E$  fibres  $\overline{X(\bar{\sigma})}$  where  $\bar{\sigma}$  J-Hölder factors of  $\bar{\sigma}(\tau)$ . Furthermore  $\overline{X(\bar{\sigma})} \neq 0 \iff \bar{\sigma} \in D(\bar{g})$ .

Analogous statements hold for  $X^\gamma(\tau)$ .

Proof: It follows from the geometric version of Breuil-Mezard conjecture in the form due to Emerton-Bee.

Genericity of  $\bar{g}$  is needed to have regular Serre weights.

Now, the crucial thm is:  $(\bar{g}: G_{F_v} \rightarrow GL_2(\mathbb{F}))$  □

Theorem 2: Suppose  $p > 3$ ,  $\bar{g}$  generic,  $M_\infty$  a fixed determinant patching functor for  $\bar{g}$ ,  $\bar{\sigma}$  a Serre weight. Then  $M_\infty(\bar{\sigma}) \neq 0 \iff \bar{\sigma} \in D(\bar{g})$ .

Proof: (a) if  $M_\infty(\bar{\sigma}) \neq 0$ , then since  $M_\infty(\bar{\sigma})$  is supported on  $\overline{X^\gamma(\bar{\sigma})}$ , hence  $\overline{X^\gamma(\bar{\sigma})} \neq 0$ . So  $\bar{\sigma} \in D(\bar{g})$  by theorem 1.

(b)  $\bar{\sigma} \in D(\bar{g})$ , let  $\tau$  be an inertial type s.t.  $D(\bar{g}) \subset JH(\bar{\sigma}^\circ(\tau))$  (proposition 1). Hence  $M_\infty(\bar{\sigma}'^\circ) \neq 0$  for some Serre wt  $\bar{\sigma}'$  so that  $M_\infty(\bar{\sigma}'^\circ(\tau)) \neq 0$  (as  $\bar{\sigma}' \in D(\bar{g}) \sim \bar{\sigma}'$  is a Jordan-Hölder factor)

By Thm 1 and the choice of  $\tau$ , it suffices to show that  $M_\infty(\bar{\sigma}'^\circ(\tau))$  is supported all of  $\overline{X^\gamma(\tau)}$

Now one uses some refinement of Theorem 2 to get  
that  ~~$X^Y(\tau)$~~  the generic fibre of  $X^Y(\tau)$  is irreducible  
 $\Rightarrow M_\infty(\sigma^\circ(\tau))$  supp. on the whole gen. fiber of  $X^Y(\tau)$ .

Cohen  $M_\infty(\overline{\sigma^\circ(\tau)})$  supp. on ~~all~~ all of  $\overline{X^Y(\tau)}$ .  
Macaulay + Taylor's argument  $\square$

Remark: From theorem 2 we retrieve Buzzard - Diamond - Jarvis conjecture by specializing our fixed determinant ~~the~~ patching functor to different contexts.

Remark 2: A priori all this depends on Gee - Kirin, Gee - Liu - Saito, but one may give different arguments for it, so that it becomes independent ~~the~~, hence giving a new proof of BDJ.