

I. Cat. of reps

Let G p -adic analytic gp, $K \in G$ open compact, L/\mathbb{Q}_p finite, $\mathcal{O} = \mathcal{O}_L$, $\bar{\omega} \in \mathcal{O}_{\text{unit}}$, $k = \mathcal{O}/\bar{\omega}$

$\text{Mod}^{\text{tors}}(\mathcal{O})$ = torsion \mathcal{O} -modules, $\text{Mod}^{\text{comp}}(\mathcal{O})$ = compact \mathcal{O} -modules

Pontryagin duality: exact anti-equivalence $\text{Mod}^{\text{tors}}(\mathcal{O}) \rightarrow \text{Mod}^{\text{comp}}(\mathcal{O})$

$$\tau \mapsto \text{Hom}(\tau, L/\mathcal{O}) = \tau^\vee$$

Let (R, m) noeth. complete local R -alg. $M^\vee = \text{Hom}(M, L/\mathcal{O}) \hookrightarrow M$

$\tau \in \text{Mod}^{\text{tors}}(\mathcal{O})$, $R \subset \mathcal{O}$ continuous with res. field k for discrete top. on $\tau \Rightarrow R \otimes \tau^\vee$ compact R -module

conversely: $M \in \text{Mod}^{\text{comp}}(\mathcal{O})$, $R \subset M$ s.t. M a compact R -mod. $\Rightarrow M^\vee$ discrete R -mod.

$\text{Mod}_G^{\text{sm}}(\mathcal{O})$ = cat. of smooth G -representations (an obj. of $\text{Mod}^{\text{tors}}(\mathcal{O})$)

similarly $\text{Mod}_K^{\text{sm}}(\mathcal{O})$; $\text{Mod}_K^{\text{fl}}(\mathcal{O}) \subseteq \text{Mod}_K^{\text{comp}}(\mathcal{O})$ - full subcat. of $\bar{\omega}$ -torsion free objects.

$\text{Mod}_K^{\text{pro}}(\mathcal{O})$ = cat. of compact $\mathcal{O}[[K]]$ -modules

$\text{Mod}_G^{\text{pro}}(\mathcal{O}) = \text{cat. of } \tau \text{ s.t. } \tau \text{ compatible } \mathcal{O}[G]\text{-action}$ $\text{Mod}_G^{\text{sm}}(\mathcal{O}) \hookrightarrow \text{Mod}_G^{\text{pro}}(\mathcal{O})$

Pontryagin duality induces exact anti-equivalences $\text{Mod}_K^{\text{sm}}(\mathcal{O}) \rightleftarrows \text{Mod}_K^{\text{pro}}(\mathcal{O})$

Let $\text{Ban}(L) = \text{cat. of Banach spaces } / L$ value gp

$\text{Ban}(L)^{\leq 1} = \tau \text{ s.t. } \|E\| \leq |L| + \text{continuous non-decreasing } L\text{-linear maps}$

Easy fact: $\text{Ban}^{\leq 1}(L)_\mathbb{Q} \cong \text{Ban}(L)$

Thm. (Schnedler - Teitelbaum) The functors $\text{Mod}_{\text{comp}}^{\text{fl}}(\mathcal{O}) \rightarrow \text{Ban}(L)^{\leq 1}$

(with top.) $E^d = \text{Hom}_G(E^\circ, \mathcal{O}) \hookrightarrow E = E^\circ = \{x \mid \|x\| \leq 1\}$
 (of pointwise converg.) $M \mapsto M^d = \text{Hom}_{\text{cont}}(M, L)$

are anti-equivalence of cats (Schildknecht duality)

Remarks on proof: (1) E^d is linear top. \mathcal{O} -module, torsion free and complete

$E^d \hookrightarrow \prod_{v \in E^\circ} \mathcal{O}$ top. embedding and $\prod_{v \in E^\circ} \mathcal{O}$ compact. so $E^d \in \text{Mod}_{\text{comp}}^{\text{fl}}(\mathcal{O})$.

(2) $M \in \text{Mod}_{\text{comp}}^{\text{fl}}(\mathcal{O}) \Rightarrow M \cong \bigoplus_{i \in I} \mathcal{O}$, some index set I .

$E \in \text{Ban}(L) \Rightarrow E \cong \zeta_0(I)$ - bounded sequences on $I = \bigoplus_{i \in I} \mathbb{Z}$ with respect to sup. norm,

(3) $X = \mathcal{O}$ -torsion free module, p -adic complete and torsion.

$X \cong \text{Hom}_{\text{cont}}(X^d, \mathcal{O})$ where $X^d = \text{Hom}(X, \mathcal{O})$ (with top. point. conv.)

Prop. (Compatibility with Pontryagin dual) Let X as above.

$$X^d / \bar{\omega}^n X^d \cong (X / \bar{\omega}^n X)^\vee$$

Let $\text{Ban}_G(L) = \text{cat. of Banach space reps of } G \text{ over } L$. $\text{Ban}_K(L) = -\text{- of } K-\text{reps}$
 $\text{Ban}_{\text{adm}}(L) = \text{adm. reps } -\text{ where } \underline{\text{admissible}} = \exists E^0 \subseteq E \text{ } K\text{-inv. } O\text{-lattice s.t.}$
 $((E/E^0)^\#)^\vee \text{ fin. gen. } O\text{-mod. } \forall H \in K \text{ open cpt}$

$\text{Mod}_{\text{comp}}^H O[K] = O\text{-torsion free compact } O[K]\text{-mod.}$

$\text{Mod}_{\text{fg}}^H O[K] = -\text{- + fin. gen. over } O[K]$.

Thm.: Schikhof duality induces anti-equivr. : $(\text{Mod}_{\text{comp}}^H O[K])_Q \xrightarrow{\sim} \text{Ban}_K(L)$
 (for G the same) $\xrightarrow{\text{continuous}}$ $(\text{Mod}_{\text{fg}}^H O[K])_Q \xrightarrow{\sim} \text{Ban}_{\text{adm}}(L)$

II. $M(\theta)$

Fix (R, m) as above and $N \in \text{Mod}_G^{\text{pro}}(\theta) + R \hookrightarrow N$ a s.t. topology on N is
 R -linear i.e. \exists basis N_i of neighb. of 0 s.t. N_i is R -modules.

For a compact R -module m , define $m \hat{\otimes} N = \varprojlim_R m/m_j \otimes_R N/N_i$.

Rem. m finitely presented $\Rightarrow m \hat{\otimes}_R N = m \otimes_R N$. basis of neighb. of 0 .

Lemma: Let $\lambda \in \text{Mod}_K^{\text{sm}}(\theta)$ of fin. length $\Rightarrow R$ -action on $\text{Hom}_K^{\text{cont}}(N, \lambda^\vee)$ is continuous

proof: $\text{Hom}_K^{\text{cont}}(N, \lambda^\vee) = \text{Hom}_K(N, \lambda, N^\vee) = \text{Hom}_K(\lambda, \varprojlim_n N^\vee[m^n])$ continuous for discrete top.

~~because $\log \lambda < +\infty$~~ $= \varprojlim_n \text{Hom}_K(\lambda, N^\vee[m^n])$ so just want ~~continuous~~.

Def. Let $\lambda \in \text{Mod}_K^{\text{sm}}(\theta)$ of fin. length, $M(\lambda) := (\text{Hom}_K^{\text{cont}}(N, \lambda^\vee))^\vee$ \square

we have $\lambda \mapsto M(\lambda)$ right-exact

(and it is left exact when N projective in $\text{Mod}_K^{\text{pro}}(\theta)$)

Prop. (i) λ as above: (ii) $k \hat{\otimes}_R N$ fin. gen. / $O[K]$ $\Rightarrow M(\lambda)$ fin. gen.

(*) (ii) let K smooth fin. gen. rep of G and assume (*) then
 $\text{Hom}_G(K, N^\vee)$ fin. gen. R -mod.

proof: (i) $M(\lambda)$ compact R -module $\overset{\text{Malayam}}{\Rightarrow}$ enough to show $M(\lambda) \hat{\otimes}_R k$ fin. dim k -v.s.
 $k \hat{\otimes}_R N$ fin. gen. $O[K]$ -mod $\Rightarrow (k \hat{\otimes}_R N)$ adm.

let $H \subseteq K$ subgroup acting trivial on λ . $((k \hat{\otimes}_R N)^\vee)^\#$ fin. dim. k -v.s.

$\Rightarrow \dim \text{Hom}_K(\lambda, (k \hat{\otimes}_R N)^\vee) < +\infty$.

$k \hat{\otimes}_R M(\lambda) = k \hat{\otimes}_R (\text{Hom}_K^{\text{cont}}(N, \lambda^\vee))^\vee = (\text{Hom}_K^{\text{cont}}(k \hat{\otimes}_R N, \lambda^\vee))^\vee$ fin. pres.

(ii) $K = \text{fin. gen.} \Rightarrow \exists c\text{-ind}_K^\theta \lambda \rightarrow K$ for some λ of fin. length.

$\sim M(\lambda^\vee) \rightarrow \text{Hom}_G(K, N^\vee)^\vee$ hence (i) \Rightarrow (ii). \square

E. Hellmann "M(θ)"

Let V be cont. K -rep. on fin. dim. L -v.s. $\Theta \in V$ K -env. O -lattice

$$\Theta^d = \text{Hom}_\Theta(\Theta, \Theta), \quad v\Theta = \text{Hom}_L(V, L).$$

Lemma: $\text{Hom}_{O[[K]]}^{\text{cont}}(N, \Theta^d)$ \bar{w} -adically complete and separated $\Rightarrow O$ -tors. free.

$$\underline{\text{proof}}: \text{Hom}_{O[[K]]}^{\text{cont}}(N, \Theta^d) \cong \varprojlim_n \text{Hom}_{O[[K]]}^{\text{cont}}(N, \Theta^d / \bar{w}^n \Theta^d)$$

on the other hand $\text{Hom}_{O[[K]]}^{\text{cont}}(N, \Theta^d) / \bar{w}^n \hookrightarrow \text{Hom}_{O[[K]]}^{\text{cont}}(N, \Theta^d / \bar{w}^n \Theta^d)$

$$\rightarrow \text{Hom}_{O[[K]]}^{\text{cont}}(N, \Theta^d) \xrightarrow{\cong} \varprojlim ? \hookrightarrow \varprojlim ?$$

Rem. R -action on $\text{Hom}_{O[[K]]}^{\text{cont}}(N, \Theta^d)$ hence \cong \square

it is cont. for discrete top. on M^d / \bar{w}^n continuous for \bar{w} -adic top. on M^d as

Def. Let $M = M(\Theta) = \text{Hom}_\Theta(\text{Hom}_{O[[K]]}^{\text{cont}}(N, \Theta^d), \Theta)$

with top. of pointwise convergence ↑ length

prop. (i) M is a compact R -module

(ii) if (*) holds then M fin. gen. $/R$.

proof: (i) as above: $\text{Hom}_{O[[K]]}^{\text{cont}}(N, \Theta^d) / \bar{w}^n$ discrete R -mod. $\Rightarrow (-)^V$ compact R -module.

$$\text{But } M \cong \varprojlim M / \bar{w}^n = \varprojlim X^d / \bar{w}^n X^d = \varprojlim (X / \bar{w}^n X)^V = (-)^V$$

$$\text{where } X = \text{Hom}_{O[[K]]}^{\text{cont}}(N, \Theta^d)$$

(ii) Nakayama \Rightarrow need to show that $k \otimes_R M$ fin. dim. $/k$. However

$$(M / \bar{w})^V \hookrightarrow \text{Hom}_{O[[K]]}^{\text{cont}}(N, \Theta^d / \bar{w}) \Rightarrow \text{Hom}_{O[[K]]}^{\text{cont}}(N, \Theta^d / \bar{w})^V \xrightarrow{\text{fin. dim.}} M / \bar{w}$$

fin. dim. k -v.s. as ↑ fin. length.

↑ fin. dim.

Prop. Assume that $d = \dim \text{supp}_{\text{spec} R} M(\Theta)$, N projective in $\text{Mod}_K^{\text{pro}}(\Theta)$ and (*).

$$\Rightarrow Z_{d-1}(M(\Theta) / \bar{w}) = \sum_\sigma m_\sigma Z_{d-1}(M(\sigma))$$

where σ runs over all isom. classes of smooth irreduc. K -reps. of K .

and $m_\sigma = \text{mult. of } \sigma \text{ in } \Theta / \bar{w}$ (as subquot.). Recall $M(\sigma) = \text{Hom}_{O[[K]]}^{\text{cont}}(N, \sigma^V)^V$.

By top. Nakayama $\Rightarrow \text{Hom}_K(\sigma, (k \otimes_R N)^V) = 0 \Rightarrow M(\sigma) = 0 \Rightarrow$ sum is finite.

proof. (i) As \bar{w} regular in $M(\Theta) \Rightarrow \dim(M / \bar{w}) = d-1$. (dim of supp.)

(ii) N proj. $\Rightarrow \text{Hom}_{O[[K]]}^{\text{cont}}(N, \Theta^d) / \bar{w} \cong \text{Hom}_{O[[K]]}^{\text{cont}}(N, \Theta^d / \bar{w})$ and reduction mod \bar{w} interchanges $(-)^d$ and $(-)^V$. Further, N proj. $\Rightarrow \lambda \mapsto M(\lambda)$ exact. and $Z_{d-1}(-)$ is additive on short exact sequences. \square

Fix m on $R[\frac{1}{p}]$ -mod. of fin. length (\Rightarrow fin. dim L -vs). Fix $m^o \subseteq m$ R -stable O -lattice $k \hat{\otimes}_R N$ fin. gen. $O[[K]]$ -module $\rightarrow m^o / \bar{\omega} \hat{\otimes}_R N$ as well.

Nakayama $\rightarrow m^o \hat{\otimes}_R N$ fin. gen. $O[[K]]$ -mod.

Define: ~~****~~ $\Pi(m) = \text{Hom}_{O[[K]]}^{\text{cont}}(m^o \hat{\otimes}_R N, L)$ adm. unitary Banach space rep.
 $m \mapsto \Pi(m)$ is left-exact (indep. of m^o if choice)

Prop. $\dim_L \text{Hom}_K(V, \Pi(m)) = \dim_L(m^o \hat{\otimes}_R M(O))$

proof: choose a presentation $R^a \xrightarrow{A} R^b \rightarrow m^o \rightarrow 0$, A matrix with coeff. in R .

$$\Rightarrow (***) 0 \rightarrow \text{Hom}_{O[[K]]}^{\text{cont}}(m^o \hat{\otimes}_R N, O^d) \xrightarrow{L} \text{Hom}_{O[[K]]}^{\text{cont}}(N, O^d)^b \rightarrow (\text{Hom} \dots)_L^a$$

as $m^o \hat{\otimes}_R N$ fin. gen. $O[[K]]$ -mod $\xleftarrow{\text{Schurhaf duality}}$ $\varphi \mapsto \varphi \cdot A$

$$\text{Hom}_{O[[K]]}^{\text{cont}}(m^o \hat{\otimes}_R N, O^d) \cong \text{Hom}_K(V, \Pi(m))$$

On the other hand $M^a \xrightarrow{A} M^b \rightarrow m^o \hat{\otimes}_R M \rightarrow 0$

$$\Rightarrow (****) 0 \rightarrow \text{Hom}_{O[[K]]}^{\text{cont}}(m^o \hat{\otimes}_R M, L) \xrightarrow{L} \text{Hom}_O^{\text{cont}}(M, L)^b \rightarrow (\dots)^a$$

$$\quad \quad \quad \gamma \mapsto \gamma \cdot A$$

now: $\text{Hom}_O^{\text{cont}}(M, L) \cong \text{Hom}_{O[[K]]}^{\text{cont}}(N, O^d) = (\text{Hom}_O^{\text{cont}}(M, L))^d$ by Schurhaf duality

$$\text{hence } (**) + (****) \Rightarrow \text{Hom}_O^{\text{cont}}(m^o \hat{\otimes}_R M, L) \cong \text{Hom}_{O[[K]]}^{\text{cont}}(m^o \hat{\otimes}_R N, L) = \\ = \text{Hom}_K(V, \Pi(m)) \quad \square$$