GDT Le Bras "Rapoport - Zink spaces I"

Fix $C/\mathbb{Q}_p$ complet, non-arch. alg. closed.

<u>Thm</u> (Scholze-Weinstein) There's an equivalence of categories:

$$\{p\text{-div}/\mathcal{O}_C\} \leftrightarrow \{(T,W) \mid T \text{ free } \mathbb{Z}_p\text{-module of finite rank},\ W \subset T \otimes_{\mathbb{Z}_p} C(-1) \text{ sub-}C\text{-vectorspace}\}$$

<u>Functor</u>: $p\text{-div}_{\mathcal{O}_C} \ni G \longmapsto (T = T(G),\ W = \mathrm{Lie}(G) \otimes C)$

lying in T via $\alpha^\vee_{G^\vee}$.

We have already shown that this functor is fully faithful and it is essentially surjective under the hypothesis that C is spherically closed and $|\cdot| : C \to \mathbb{R}^+$ is surjective.

§1. Connections between p-div gps and vector bundles on the FF curve

$B^+_{cris} = B^+_{cris}(\mathcal{O}_C/p)$, $P = \bigoplus_{d \geq 0} (B^+_{cris})^{\varphi = p^d}$, $X = \mathrm{Proj}\ P$.

if G_0 is p-div gp over $\mathcal{O}_C/p$, M Dieudonné module of G_0.

Let $\mathcal{E}(G_0)$ be the vector bundle on X ass. to graded P-module $\bigoplus_{d \geq 0} (M[p^{-1}])^{\varphi = p^{d+1}}$

<u>Thm.</u> Functor $G_0 \longmapsto \mathcal{E}(G_0)$ is fully faithful with essential image being vector bundles on X of slopes between 0 and 1.

<u>Prop.</u> Let $G \in p\mathrm{div}_{\mathcal{O}_C}$, $\mathcal{E} = \mathcal{E}(G_0)$, $G_0 = G \otimes_{\mathcal{O}_C} \mathcal{O}_C/p$, $\mathcal{F} = \mathcal{O}_X \otimes_{\mathbb{Z}_p} T$ $(T = T(G)(\mathcal{O}_C))$

Then: (1) There exists an exact sequence

$$0 \to \mathcal{F} \to \mathcal{E} \to i_{\infty *}(\mathrm{Lie}\, G \otimes C) \to 0$$ with global sections

$$0 \to T[\tfrac{1}{p}] \to \tilde{G}(\mathcal{O}_C) \to \mathrm{Lie}\, G \otimes C \to 0$$ (where $\infty \in X$ is a distinguished point on X)

(2) If we identify $i^*_\infty \mathcal{E} = M(G) \otimes_{\mathcal{O}_C} C$, then the global sections of $\mathcal{E} \to i_{\infty *} i^*_\infty \mathcal{E}$ give the quasi-logarithm $q\log : \tilde{G}(\mathcal{O}_C) \to M(G) \otimes C$ restricted to $T[\frac{1}{p}]$ give $T \otimes C \xrightarrow{\alpha_G} (\mathrm{Lie}\, G^\vee \otimes C)^\vee$.

<u>Summary</u>: to give a p-div group over $\mathcal{O}_C$ is the same as to give a modification $\mathcal{E}$ of trivial vector bundle $\mathcal{O}_X^h$ along $\infty \in X$.

§2. Rapoport-Zink spaces (level 0)

H p-div gp over a perfect field k of char p, height h, dim d.

<u>Def.</u> $R \in \mathrm{Nilp}_{W(k)}$: deformation of H to R is a pair (G, ρ) s.t. G is a p-div gp over R, $\rho : H \otimes_k R/p \to G \otimes_R R/p$ quasi-isogeny

we define $\mathrm{Def}_H : R \longmapsto \{$isomor. classes of deformations of H to $R\}$

Thm. (Rapoport-Zink) Functor Def_H is representable by a formal scheme $\mathcal{M}$ over $\mathrm{Spf}(W(k))$ which locally has an ideal of definition of finite type, and irred. components of a reduced scheme ass. are proper. $\mathcal{M} = \varinjlim \mathcal{M}^n$.

Period morphism: $M(H)$ – $W(k)$-module, free of rank h

R – $W(k)$-algebra, complete for p-adic top.

Grothendieck-Messing: $(G,\rho) \longmapsto M(H)\otimes R[\frac{1}{p}] \twoheadrightarrow \mathrm{Lie}(G)[\frac{1}{p}]$

depends on (G,ρ) only up to quasi-isog.

$\leadsto$ $\pi : \mathcal{M}^{ad}_{\eta} \to \mathrm{Grass}(d,h)$ ← param. quotients of $M(H)[\frac{1}{p}]$ of dim d.

period morphism

We can also look at $(\mathrm{Def}_H^{isog}) = R$ flat, complete $W(k)$-alg. $\mapsto \{(G,\rho)\}/$quasi-isog.

After remark above, π factorises via $\pi : (\mathrm{Def}_H^{isog})^{ad}_{\eta} \to \mathrm{Grass}(d,h)$

Prop. Functor $(\mathrm{Def}_H^{isog})^{ad}_{\eta}$ is representable by an adic space, which can be identified via π with an open $\mathcal{U} \subset \mathrm{Grass}(d,h)$.

Proof: $\mathcal{U}$ = image of π, π étale $\Rightarrow$ $\mathcal{U}$ open.

① show that for all $(W(k)[\frac{1}{p}], W(k))$-algebras, affinoid, complete (R,R^+) map $(\mathrm{Def}_H^{isog})^{ad}_{\eta}(R,R^+) \to \mathcal{U}(R,R^+)$ is injective.

If we choose two elts with the same image $(G_i,\rho_i)/R_0 \subset R^+$ open, bounded, $i=1,2$.

quasi-isog. $\rho_i : H\otimes_k R_0/p \to G_i \otimes_{R_0} R_0/p$.

Gr.-Mess. $\Rightarrow$ obstruction to lift $\rho_2\circ\rho_1^{-1}$ (q. isog. of $G_1\otimes R_0/p$ and $G_2\otimes R_0/p$) to a quasi-isog. of G_1 and G_2 is given by the compatibility with the Hodge filtration. $\xRightarrow[\text{by } \pi]{\text{images}}$ quasi-isogenous so ok.

② **Lemma**: $\pi : \mathcal{M}^{ad}_{\eta} \to \mathcal{U}$ has local sections.

proof: It suffices to show that $\forall$ field K non-arch complete $/\mathbb{Q}_p$, K^+ val. ring. ($\mathcal{M}^{ad}_{\eta}(K,K^+) \to \mathcal{U}(K,K^+)$). Then $u\in \mathcal{U}(K,K^+)$ lifts to an neighb. of (K,K^+) of $\mathcal{M}^{ad}_{\eta}$.

We can suppose $K^+ = \mathcal{O}_K$.

Let $x \in \mathcal{U}(K,\mathcal{O}_K)$, $x\in$ image (π) $\Rightarrow$ $\exists_{L/K}$ finite Galois and $\tilde{x}\in \mathcal{M}^{ad}_{\eta}(L,\mathcal{O}_L)$ which lifts x. $\Leftrightarrow$ $G\in$ pdiv$/\mathcal{O}_L$ and q. isog. ρ

$\leadsto$ $T(G) = T(G)(\mathcal{O}_{\bar{K}})$ has an action of $Gal(\bar{K}/L)$.

Claim: $V(G)$ has an action of $Gal(\bar{K}/K)$.

In fact, if $\sigma \in Gal(\bar{K}/K)$, then $\sigma(\tilde{x})$ projects on x by π.

Injectivity: quasi-isog. between $\sigma^*(G)$ and G compatible with $\sigma^*(\rho)$ and ρ.

$\Rightarrow$ canonical identification of $V(G)$ and $\sigma^*(V(G))$ $(\sigma \curvearrowright V(G))$

$Gal(\bar{K}/K)$ acts on $V(G)$ $\rightsquigarrow$ $W(k)$-lattice T has an action of Gal.

$\Rightarrow$ ∃ p-div gp. G'/O_L quasi-isogeneous to G. $\Rightarrow x' \in \mathcal{M}^{ad}_\eta(K, K^+) \to x$ □

§3 Image of period morphism

$H/\bar{\mathbb{F}}_p$ p-div gp. ht h, dim d. $\mathcal{M}$ - Rapoport-Zink space, $\pi : \mathcal{M}^{ad}_\eta \to Gross(d,h)$

$C/\mathbb{Q}_p$ complet, alg. closed; $x \in Gross(d,h)(C, O_C) \hookrightarrow$ quotient of dim d $M(H) \otimes C \to W$.

$\mathcal{E} = \mathcal{E}(H)$; $\mathcal{E} \to i_{\infty *} i_\infty^* \mathcal{E} = i_{\infty *}(M(H) \otimes C) \to i_{\infty *}(W)$

$$\Rightarrow \quad 0 \to \mathcal{F} \to \mathcal{E} \to i_{\infty,*} W \to 0 . \quad (\mathcal{F} = \text{kernel of the above}).$$

Thm. $x \in \text{image}(\pi) \Leftrightarrow \mathcal{F} \cong O_X^h$.

Proof: It suffices to prove it for C spherically closed with $|.|: C \to \mathbb{R}$ surjective because π is locally of finite type and because the condition on $\mathcal{F}$ is invariant by ext. of scalars of base field (by classification of bundles on X).

Let $\mathcal{F}$ be trivial
Now, we choose $\mathbb{Z}_p$-lattice T in $H^0(X, \mathcal{F}) = \mathbb{Q}_p^h$

Modification $\Rightarrow W \hookrightarrow T \otimes C(-1)$. Case of class. thm gives

$G \in pdiv/O_C$ s.t. its associated modification is $0 \to \mathcal{F} \to \mathcal{E} \to i_{\infty *} W \to 0$

We have: $\mathcal{E}(H) = \mathcal{E}(H \otimes_{\bar{\mathbb{F}}_p} O_C/p) \cong \mathcal{E}(G \otimes_{O_C} O_C/p)$. So we want to find a quasi-isogeny $\rho: H \otimes_{\bar{\mathbb{F}}_p} O_C/p \to G \otimes_{O_C} O_C/p$. This is given by fully faithfulness of Dieudonné module. □

Proof of classification thm for general C: Let (T, W) be as in thm.

We construct

$$\begin{array}{c} 0 \to \mathcal{F} \to \mathcal{E} \to i_{\infty *} W \to 0 \\ \| \\ T \otimes_{\mathbb{Z}_p} O_X \end{array}$$

Claim: $\exists H \in p\text{-div}/\bar{\mathbb{F}}_p$ s.t. $\mathcal{E} = \mathcal{E}(H)$.

Admitting $\mathcal{H}_*$: $0 \to \mathcal{F} \to \mathcal{E}(H) \to i_{\infty,*} W \to 0$ gives a quotient $M(H)\otimes C \to W$

hence a $x \in \mathrm{Grass}(d,h)$, $\mathcal{F}$ trivial $\rightsquigarrow \tilde{x}$ pt ~~of~~ RZ space $\Rightarrow$

$\Rightarrow G' \in p\text{-div}/\mathcal{O}_C$, quasi-isog. $\rho : H\otimes \mathcal{O}_C/p \to G \otimes \mathcal{O}_C/p$.

$T(G') \subset H^0(X,\mathcal{F}) = T[\frac{1}{p}]$ is a $\mathbb{Z}_p$-lattice. Up to replacing G' by a quasi-isog. G we can suppose $T(G) = T$. Hence we are left with proving the Claim.

Let $\mathcal{O}_X(\lambda)$ be a direct summand of $\mathcal{E}$. We have a non-zero mor. $\mathcal{O}_X \to \mathcal{O}_X(\lambda)$. If $\mathcal{F} \not\cong \mathcal{O}_X^h$ then $\mathcal{O}_X(\lambda) \twoheadrightarrow \underbrace{\mathrm{coker}(\mathcal{F} \to \mathcal{E})}_{\text{skyscraper sheaf}}$ so $\lambda \geq 0$. Look at:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d] & i_{\infty *} W \arrow[d] \\ 0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{E}' \arrow[r] & i_{\infty *}(T(G)\otimes C(-1)) \end{tikzcd}$$

where $G' = T(G)(-1) \otimes \mu_{p^\infty}$, $\mathcal{E} = T(G) \otimes_{\mathbb{Z}_p} \mathcal{O}_X(1)$.

non-zero map $\mathcal{O}_X(\lambda) \to \mathcal{O}_X(1) \Rightarrow \lambda \leq 1$ hence the proof (by isotriviality) $\square$

§.4 RZ spaces at ∞

$(W(k)[\frac{1}{p}], W(k))$ - complete aff. alg.

$H \in p\text{div}/k$. **Def.** $\mathcal{M}_n : (R,R^+) \longmapsto (G,\rho,\alpha)$ with $(G,\rho) \in \mathcal{M}_\eta^{ad}(R,R^+)$ and $\alpha : (\mathbb{Z}/p^n\mathbb{Z})^h \to G[p^n]_\eta^{ad}(R,R^+)$ of $(\mathbb{Z}/p^n\mathbb{Z})$-modules.

such that if $x = \mathrm{Spa}(K,K^+) \in \mathrm{Spa}(R,R^+)$: $\alpha(x) : (\mathbb{Z}/p^n\mathbb{Z})^h \to G[p^n]_\eta^{ad}(K,K^+)$ isom.

The same def. for $\mathcal{M}_\infty$ with $\alpha : \mathbb{Z}_p^h \to T(G)(R,R^+)$

Prop. (R.Z) $\mathcal{M}_n$ is repr. by an adic space over $\mathrm{Spa}(W(k)[\frac{1}{p}], W(k))$ open and closed of $(G[p^n]_\eta^{ad})^h$ - fiber product of $G[p^n]_\eta^{ad}$ with itself over $\mathcal{M}_\eta^{ad}$.

universal p-div group

Thm. (S.W): $\mathcal{M}_\infty$ is rep. by an adic space which preperfectoid (adic space with open covering $\mathrm{Spa}(A,A^+)$ s.t. $(\hat{A},\hat{A}^+)$ (compl. for p-adic top.) is affinoid perfectoid)

Idea of proof: for representability of $\mathcal{M}_\infty$, look at:

$$\begin{tikzcd} \mathcal{M}_\infty \arrow[r] \arrow[d] & (T(G)_\eta^{ad})^h \arrow[d] \\ \mathcal{M}_1 \arrow[r] & (G[p]_\eta^{ad})^h \end{tikzcd}$$

which is a cartesian diagram.

This gives representability of $\mathcal{M}_\infty$.

We have $\mathcal{M}_\infty \sim \varprojlim \mathcal{M}_n$

in sense of adic spaces!

$\mathcal{M}_\infty$ preperfectoid : describe it differently; take universal cover $\tilde{H} \Rightarrow$

$\Rightarrow \tilde{H}^{ad}_\eta$ / $Spa(W(k)[\frac{1}{p}], W(k))$; $q\log : \tilde{H}^{ad}_\eta \to M(H) \otimes \mathbb{G}_a$

one proves that $\mathcal{M}'_\infty : (R, R^+) \mapsto (s_1, \dots, s_h) \in \tilde{H}^{ad}_\eta (R, R^+)^h$ s.t.

(1) $(q\log(s_1), \dots, q\log(s_h))$ have rank $h-d$.

(2) $(s_1(x), \dots, s_h(x))$ $\mathbb{Q}_p$-lin. indep. for all $x = Spa(K, K^+) \in Spa(R, R^+)$.

is equal to $\mathcal{M}_\infty$ i.e. $\mathcal{M}'_\infty \cong \mathcal{M}_\infty$ and $\mathcal{M}'_\infty$ is subfunctor locally closed of $(\tilde{H}^{ad}_\eta)^h$.

It is basically done by constructing:

(i) $\mathcal{M}_\infty \to \mathcal{M}'_\infty$

$\alpha : \mathbb{Z}_p^h \to \tilde{H}^{ad}_\eta (R, R^+) \Rightarrow$ gives sections $s_1, \dots, s_h$

(ii) harder one is $\mathcal{M}'_\infty \to \mathcal{M}_\infty$. Prove:

① $\mathcal{M}'_\infty \to Gress(d,h)$ factorises via $\mathcal{U}$. Proof: $s_1, \dots, s_h$ gives $\mathcal{O}_X^h \to \mathcal{E} = \mathcal{E}(H)$

Let W = quotient of $M(H) \otimes C$ by C-v.sp. gen. by $(s_1, \dots, s_h)$

$\rightsquigarrow$ $\mathcal{O}_X^h = \mathcal{F} \xrightarrow{\sim} \mathcal{F}' = \ker(\mathcal{E} \to i_{\infty *} W)$

one proves it by using X.

Hence $0 \to \mathcal{O}_X^h \to \mathcal{E}(H) \to i_\infty W \to 0$ which gives ①

use ① to construct $\mathcal{M}'_\infty \to \mathcal{M}_\infty$ finishing the proof. $\square$