P. Chojecki GDT "Kisin's theory on the curve" 1

§. 1. Adic structure on the curve

Notation: E local field (I'll assume $\mathrm{char}\, E = 0$ for simplicity; i.e. $E/\mathbb{Q}_p$ finite)

$\mathbb{F}_q = \mathcal{O}_E/(\pi)$, π unif.; $F/\mathbb{F}_q$ perfect, complete for $v: F \to \mathbb{R} \cup \{\infty\}$.

Let $\mathcal{E} = W_{\mathcal{O}_E}(F)[\frac{1}{\pi}] = \{ \sum_{n \gg -\infty} [x_n]\pi^n \mid x_n \in F \}$: $\mathcal{O}_{\mathcal{E}}/\pi\mathcal{O}_{\mathcal{E}} = F$

for an interval $I \subset]0,1[$ put

B_I = completion of $B^b = \{ \sum_{n \gg -\infty} [x_n]\pi^n \in \mathcal{E} \mid \exists_C, \forall_n \ |x_n| \leq C \}$

for a family of normes $(|\cdot|_\rho)_{\rho \in I}$, where we put

$|x|_\rho = q^{-v_r(x)}$ where $v_r(x) = \inf_{n \in \mathbb{Z}} \{ v(x_n) + nr \}$ if $\rho = q^{-r}$.

Put $B = B_{]0,1[} := \varprojlim_{I \subset]0,1[\text{ compact}} B_I$

φ there is an action of φ on it (Frobenius)

Let $|Y|$:= maximal closed ideals of B = ideals generated by primitive elts $(:= \{ \sum_{n \geq 0} [x_n]\pi^n \mid x_n \in \mathcal{O}_F, x_0 \neq 0, \exists_d \ x_d \in \mathcal{O}_F^\times \})$

For $\mathfrak{m} \in |Y|$ put $L_\mathfrak{m} = B/\mathfrak{m}$ = perfectoid field s.t. its tilt $L_\mathfrak{m}^b$ is an extension of F.

if $F = \bar{F}$ then $\bar{L}_\mathfrak{m} = L_\mathfrak{m}$ and $\mathfrak{m} = (\pi - [a])$ for some $a \in \mathfrak{m}_F \setminus \{0\}$

Put $P = \bigoplus_{d \geq 0} B^{\varphi = p^d}$ and define the curve

$$X = \mathrm{Proj}\, P$$

One can show that its residual fields at closed points are perfectoid. We also can show that $|Y|/\varphi^\mathbb{Z} \xrightarrow{\sim} |X|$. This is helpful as we want to define X^{ad} i.e. ~~an~~ a structure of an adic space on X and we will use for it $Y^{ad}/\varphi^\mathbb{Z}$.

Fargues-Fontaine prove that $\forall_{I \subset]0,1[}$ B_I is Banach algebra which is a principal ideal ring.

We put $Y_I^{ad} = \mathrm{Spa}(B_I, B_I^\circ)$ then one shows:

<u>thm.</u> Y_I^{ad} is an adic space, i.e. a presheaf $\mathcal{O}_{Y_I^{ad}}$ is a sheaf.

<u>proof</u>: there are two steps

1) put $E_\infty = \bigcup_{n \geq 0} E(\pi^{1/p^n})$, then $B_I \hat{\otimes}_E \hat{E}_\infty$ is perfectoid algebra (hence it defines an adic space)

2) descend the result to E by using $\hat{\bar{E}}^{Gal(\bar{E}/E)} = E$ □

In fact, (FF) conjecture that B_I strongly noetherian that is $\forall_n$ $B_I\langle X_1, \dots, X_n\rangle$ is noetherian

(<u>application</u>: definition of $X^{ad} \times_E \mathrm{Spa}(A, A^\circ)$ for A an affinoid E-algebra ~~affinoid~~. which can be used in the study of Galois reps. with coeffs over A)

then we have for $I \subset I' \Rightarrow Y_I^{ad} \subset Y_{I'}^{ad}$ (open, rational domain)

and we put $Y^{ad} = \varinjlim_{I \subset]0,1[} Y_I^{ad}$ - it's an adic space $/E$.

we have $|Y| \subset |Y^{ad}|$ - $|Y|$ can be seen as "classical points"

after all we put $X^{ad} = Y^{ad}/\varphi^{\mathbb{Z}}$.

<u>§.2 Vector bundles</u>

Suppose $F = \bar{F}$, $\lambda \in \mathbb{Q}$ and put $\mathcal{O}_X(\lambda)$ = vector bundle on X of slope λ (it's constructed "explicitely")

<u>thm.</u> We have

$$\{\lambda_1 \geq \dots \geq \lambda_n \mid n \in \mathbb{N}, \lambda_i \in \mathbb{Q}\} \xrightarrow{\sim} \mathrm{Bun}_X/\sim$$
$$(\lambda_i)_i \longmapsto [\oplus_i \mathcal{O}_X(\lambda_i)]$$

For any F, we have to add an action of $G_F = Gal(\bar{E}/F)$ and one proves that

$$\mathrm{Bun}_{X_F} \xrightarrow{\sim} \mathrm{Bun}_{X_{\hat{\bar{F}}}}^{G_F}$$

<u>where</u>: X_F = the curve for F

$X_{\hat{\bar{F}}}$ = -"- $\hat{\bar{F}}$

G_F-equivariant vector bundles on $X_{\hat{\bar{F}}}$ (+ continuity + technical cond.)

Define the Robba ring by $\mathcal{R}_F = \varinjlim_{\varrho \to 0} B_{]0,\varrho]}$

we have $\forall_{\varrho \in]0,1[}$ $Y^{ad} = \bigcup_{n \geq 0} \varphi^{-n}(Y^{ad}_{]0,\varrho]})$ and hence

$$\mathrm{Bun}_{X^{ad}} \xrightarrow{\sim} \mathrm{Bun}^{\varphi}_{Y^{ad}} \xrightarrow{\sim} \varphi\text{-}\mathrm{Mod}_{\mathcal{R}_F}$$

$\uparrow$ φ-equivariant bundles

Using Kedlaya classification of φ-$\mathrm{Mod}_{\mathcal{R}_F}$ (i.e. slope decomposition for φ-modules) we obtain, by comparison with the result of Fargues-Fontaine, a form of GAGA theorem:

$$\mathrm{Coh}_X \xrightarrow{\sim} \mathrm{Coh}_{X^{ad}}$$
$$\mathcal{F} \longmapsto \mathcal{F}^{ad}.$$

§.3 Kisin's theory on the curve

Take $F = \bar{F}$, put $\mathfrak{S} = W_{\mathcal{O}_E}(\mathcal{O}_F) = \{ \sum [x_n] \pi^n \mid x_n \in \mathcal{O}_F \}$ with φ Frob.

Define $\boxed{\mathrm{Mod}^{\varphi}_{\mathfrak{S}}}$ = free $\mathfrak{S}$-modules M of finite rank + a map $\varphi_M : M \to M$ which is φ-linear and s.t. coker φ is killed by some primitive elt (i.e. $\sum_{n \geq 0} [x_n] \pi^n$, $x_0 \neq 0$, $\exists d : x_d \in \mathcal{O}_F^{\times}$).

$\boxed{\mathrm{Modif}}$ = category of modifications $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{F} \to 0$ where $\mathcal{E}', \mathcal{E} \in \mathrm{Bun}_X$, $\mathcal{F} \in \mathrm{Coh}^{tor}_X$ (torsion) $\mathcal{E}'$ trivial v.b.

$(M, \varphi_M) \in \mathrm{Mod}^{\varphi}_{\mathfrak{S}} \rightsquigarrow \exists \mathcal{G} \in \mathrm{Coh}^{tor}_{Y^{ad}}$ (torsion) s.t. $H^0(Y^{ad}, \mathcal{G}) = \mathrm{coker}\, \varphi_M$.

$$\rightsquigarrow \mathcal{F}(M,\varphi) = \bigoplus_{n \in \mathbb{Z}} \varphi^{n*} \mathcal{G} \in \mathrm{Coh}^{tor}_{Y^{ad}/\varphi^{\mathbb{Z}}} = \mathrm{Coh}^{tor}_X.$$

define also an isocrystal $(D,\varphi) = (M,\varphi) \otimes_{W_{\mathcal{O}_E}(\mathcal{O}_F)} W_{\mathcal{O}_E}(k)[\frac{1}{\pi}]$ (k-Isocrystal where k = res. field of $\mathcal{O}_F$)

and put $\mathcal{E}(M,\varphi) := \mathcal{E}(D,\varphi) = \bigoplus_{\lambda \in \mathbb{Q}} \mathcal{O}_X(\lambda)^{m_\lambda} \in \mathrm{Bun}_X$ (m_λ = multiplicity of (D,φ) isocrystal (i.e. Dieudonné-Manin decomp.)

put also $\mathcal{E}'(M,\varphi) = \widetilde{\left(\bigoplus_{d \geq 0} \mathrm{Hom}_{\mathfrak{S}}(M, B)^{\varphi = \pi^d} \right)} \in \mathrm{QCoh}_X$

So we have a functor and a conjecture:

<u>Conj.</u> There exists a contravariant essentially surjective functor:

$$\mathrm{Mod}^{\varphi}_{\mathfrak{S}} \longrightarrow \mathrm{Modif}$$
$$(M,\varphi) \longmapsto [0 \to \mathcal{E}'(M,\varphi) \to \mathcal{E}(M,\varphi) \to \mathcal{F}(M,\varphi) \to 0]$$

This is related to p-divisible groups. Take $\infty \in |X|$ and $C = k(\infty)$. Then:

<u>Conj.</u> Let $m \in |Y|$ s.t. $m \mapsto \infty$ via $|Y|/\varphi^{\mathbb{Z}} \xrightarrow{\sim} |X|$, $B/m = C$. There exists an (anti) equivalence of categories:

$$\mathrm{pdiv}_{\mathcal{O}_C} \simeq \{ (M,\varphi) \in \mathrm{Mod}^{\varphi}_{\mathfrak{S}} \mid \mathrm{coker}\, \varphi \text{ is killed by } m \}.$$

It would be interesting to compare it with the description of $\mathrm{pdiv}_{\mathcal{O}_C}$ given by Scholze and Weinstein, i.e.

<u>Thm.</u> (Scholze-Weinstein): There is an equivalence of categories:

$$\mathrm{pdiv}_{\mathcal{O}_C} \simeq \left\{ (T,W) \;:\; \begin{array}{l} T \text{ - free } \mathbb{Z}_p\text{-module of finite rank} \\ W \subset T \otimes C(-1) \text{ - } C\text{-subvectorspace} \end{array} \right\}$$