GDT Chojecki "p-div$_{O_C}$" 

Let $C/\mathbb{Q}_p$ be an alg. closed, complete field. We want to prove

Thm (Scholze-Weinstein) $\{\text{p-divisible groups}/O_C\} \cong \{(T,W) \mid T \text{ finite free } \mathbb{Z}_p\text{-mod.}, W \subseteq T\otimes_{\mathbb{Z}_p} C(1), \text{ C-sub vector space}\}$

In this talk we'll the first step toward it, namely we'll prove that the Dieudonné module is fully faithful in certain cases.

## §.1 Preparation on p-divisible groups

Recall that a p-div group $G=(G_n)_n$ is a f.p.p.f sheaf s.t.

$0\to G_n \to G_{n+1} \xrightarrow{[p^n]} G_{n+1} \to 0$ is exact.

examples. (1) $(\mathbb{Q}_p/\mathbb{Z}_p) = \left(\frac{\frac{1}{p^n}\mathbb{Z}}{\mathbb{Z}}\right)_n$ (2) $\mu_{p^\infty} = \left(\ker(\mathbb{G}_m \xrightarrow{[p^n]} \mathbb{G}_m)\right)_n$.

Def: $f: G\to G'$ is an isogeny if it is f.p.p.f. epimorphism with finite loc.free kernel.

A quasi-isogeny $G \rightsquigarrow G'$ is a global section $g$ of the (Zariski) sheaf $\underline{\mathrm{Hom}}_S(G,G')\otimes_{\mathbb{Z}}\mathbb{Q}$ s.t. locally $\exists_n$ $p^n g$ is an isogeny. ($G, G'$ are p-div. over $S$)

Def. $S$ scheme (adic space), $M$ quasi-coh $O_S$ sheaf $\rightsquigarrow \widehat{M}$ an fppf $S$-group s.t. $\widehat{M}(T) = O(T)\otimes_{O_S(T)} M(T)$ for $S$-spaces $T$.

If $M$ is loc. free $O_S$-module of finite rank then $\widehat{M}$ is representable by a group scheme loc. iso. to a finite product of $\mathbb{G}_a$. In this case, we will call $\widehat{M}$ a vector group. (over $S$).

Let $R$ be a p-torsion ring and let $G$ be a p-div. gp. over $R$.
Among all the extensions $0\to V\to E\to G\to 0$ (i.e. exact seq. of comm. $R$-groups) there exists ~~a~~ the universal one: $0\to V \to EG\to G\to 0$.

We will write $M(G)$ for $\mathrm{Lie}\, EG$ (it is a crystal).

In this context, one shows that $(S,S^+)\mapsto M\otimes_R S$ (where $M$ is locally free $R$-module) is representable by an adic space over $\mathrm{Spa}(R,R)_\eta$ (i.e. $\eta = \ldots \times_{\mathrm{Spa}(R,R)} \mathrm{Spa}(K,R)$) which we denote by $M\otimes \mathbb{G}_a$ (it was $\widehat{M}$ above)

Let $T(G)(S) = \varprojlim_n G[p^n](S)$ be the Tate module. It is representable over $R$ and we have $T(G)^{ad}_\eta \sim \varprojlim G[p^n]^{ad}_\eta$

notion in the context of adic spaces

## §2. Dieudonné theory over semiperfect rings

Let $R$ be a ring of characteristic $p$.

Def. A ring $R$ is <u>semiperfect</u> if the Frobenius $\Phi: R \to R$ is surjective.

A map $f: R \to S$ of semiperfect rings is ~~an~~ <u>isogeny</u> if $I = \ker(f)$ satisfies $\Phi^n(I) = 0$ for some $n$.

Notion of being isogenous gives an equivalence relation on semiperfect rings.

~~Rem~~ Prop. Let $R$ semiperfect, $R^\flat = \varprojlim_{\Phi} R$ perfection (tilt). Then the following are equivalent:

(i) $R^\flat$ is $f$-adic (i.e. has finitely generated ideal of definition)

(ii) $R$ is isogenous to a semiperfect ring $S$ which is the quotient of a perfect ring $T$ by a finitely generated ideal $J \subset T$.

Def. ~~If~~ $R$ is <u>$f$-semiperfect</u> if $R$ is semiperfect and $R^\flat$ is $f$-adic. In this case, $R$ is isogenous to a quotient of a perfect ring $T$ by an ideal $J \subset T$ s.t. $\Phi(J) = J^p$.

We use also

Prop. (Fontaine). Let $R$ semiperfect. Then there exist a universal $p$-adically complete PD thickening $A_{cris}(R)$ of $R$. The construction of $A_{cris}(R)$ is functorial in $R$, in particular, there is a Frobenius $\varphi$ on $A_{cris}(R)$.

Proof: $A_{cris}(R) = p$-adic completion of PD hull of $W(R^\flat) \to R$.

<u>Dieudonné functor</u>: $p\text{-div}_R \ni G \longmapsto M = M(G)(A_{cris}(R))$ finite, projective $A_{cris}(R)$-module.

with maps $F: M \otimes_{A_{cris}(R),\varphi} A_{cris}(R) \to M$, $V: M \to M \otimes_{A_{cris}(R),\varphi} A_{cris}(R)$

s.t. $FV = VF = p$

We can pass with this functor to the cat. of $p$-div grps up to isogeny getting $B^+_{cris}(R) = A_{cris}(R)[\frac{1}{p}]$-modules with $F, V$ as above.

<u>Main Theorem (Scholze-Weinstein)</u> Let $R$ be an $f$-semiperfect ring. Then the Dieudonné functor on $p$-divisible groups up to isogeny is fully faithful.

<u>Sketch of a proof</u>: (a) One proves it firstly for $\mathrm{Hom}_R(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty})$ by an explicit computation. In this case $M(\mathbb{Q}_p/\mathbb{Z}_p)$, $M(\mu_{p^\infty})$ are free $A_{cris}(R)$-modules of rank one, with $F$ acting as $p$ (resp. $1$) on $M(\mathbb{Q}_p/\mathbb{Z}_p)$ (resp. $M(\mu_{p^\infty})$)

(b) for general morphism $G \to H$ one uses base change to the ring with universal homomorphisms $\mathbb{Q}_p/\mathbb{Z}_p \to G$ and $H \to \mu_{p^\infty}$ hence a map $\mathbb{Q}_p/\mathbb{Z}_p \to G \to H \to \mu_{p^\infty}$ and we use the special case.

## §3. the case $\mathbb{Q}_p/\mathbb{Z}_p \to \mu_{p^\infty}$.

Let $J = \ker(R^b \to R)$. We have

**lemma**: $\mathrm{Hom}_R(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty}) = 1 + J \subset R^b$ under which the canonical map: $\mathrm{Hom}_R(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty}) \to \mathrm{Hom}(M(\mathbb{Q}_p/\mathbb{Z}_p), M(\mu_{p^\infty}))[\frac{1}{p}] = B^+_{cris}(R)^{\varphi = p}$ is identified with the map $1+J \longrightarrow B^+_{cris}(R)^{\varphi=p} : \varepsilon \longmapsto \log([\varepsilon])$

**Proof**: easy computation + lemma below. □

**lemma**: The map: $\mathrm{Hom}_R(\mathbb{Q}_p/\mathbb{Z}_p, G)[\frac{1}{p}] \to \mathrm{Hom}(M(\mathbb{Q}_p/\mathbb{Z}_p), M(G))[\frac{1}{p}] \hookrightarrow M(G)[\frac{1}{p}]$ given by evaluation at $1 \in M(\mathbb{Q}_p/\mathbb{Z}_p)$ agrees with the map $q^{\log} : \widetilde{G}(R) = \varprojlim_{x \mapsto px} G(R) \longrightarrow M(G)[\frac{1}{p}]$.

where $q\log$ is the composite $\widetilde{G} \xrightarrow{s_G} EG \xrightarrow{\log_{EG}} M(G) = \mathrm{Lie}\, EG$ with $s_G$ explicit. ($s_G(x) = \lim_{n \to \infty} p^n y_n$ where $y_n$ lift of $(x_n) = x \in \widetilde{G}(R)$)

We want to prove that $1+J \to B^+_{cris}(R)^{\varphi=p}$ as above is <u>bijective</u>.

**injectivity**: let $I = \ker(W(R^b) \to R)$. It follows from

**lemma**: let $w \in 1+I$. If $\log w = 0$ then $w = 1$ (assume $p \neq 2$ for simplicity)

**proof**: algebraic manipulations. □

**surjectivity**: let $I_{cris}(R) \subset A_{cris}(R)$ be the kernel of $A_{cris}(R) \to R$.

**lemma**: there is a unique $\varphi$-linear map $\varphi^1 : I_{cris}(R) \to A_{cris}(R)$ such that $p\varphi^1 = \varphi$, $\varphi^1(p) = 1$ and $\varphi^1(\gamma_n([x])) = (p-1)!\binom{np}{p}\gamma_{np}([x])$ for all $x \in J$ and $n \geq 1$

PD structure (i.e. "$\frac{x^n}{n!}$")

**remark**: this is a model for "$\frac{\varphi}{p}$".

Put $[J] := \{ \sum_{i \geq 0} [r_i] p^i \in W(R^b) \mid r_i \in J \} \subset W(R^b)$

and let $N = \ker( A_{cris}(R) \to W(R^b)/[J] )$ (we omit the construction of this map)

Take any $a \in B^+_{cris}(R)^{\varphi = p}$. We want to find $r \in 1+J$ s.t. $a = \log([r])$.

we can assume (multiply "$a$" by a power of $p$) that

$a \in I_{cris}(R)^{\varphi^1 = 1}$. As $W(R^b) \twoheadrightarrow W(R^b)/[J]$ is surjective we can

write $a = w + n$. with $n \in N$ and $w \in W(R^b)$ (assume $w \in p^2 W(R^b)$)

consider $z = \varphi^1(w) - w = n - \varphi^1(n) \in [J]$

write $z = \sum_{i \geq 0} [z_i] p^i$ ($z_i \in J$ $\forall i$) and observe that

$$a = w + n = w + z + \varphi^1(z) + (\varphi^1)^2(z) + \dots + (\varphi^1)^k(z) + \dots$$

(one proves that $\varphi^1_{|N}$ is topologically nilpotent so it makes sense)

recall the Artin-Hasse exponential series:

$$AH(t) = \exp\left(t + \frac{t^p}{p} + \frac{t^{p^2}}{p^2} + \dots\right) \in \mathbb{Z}_p[[t]].$$

one then proves that $AH([z_i])$ converges to an element in $1+I$.

We put $\xi = \exp(w) \prod_{i \geq 0} AH([z_i])^{p^i}$.

which converges also to an element of $1+I$.

Then $\log \xi$ converges in $A_{cris}(R)$ to $a = w+n$.

Let $r$ be the image of $\xi$ in $R^b$, so $r \in 1+J$. Then

one shows that $\xi = [r]$ and thus $a = \log([r])$ as wanted $\square$

## §4. General case

Recall the Tate module $TG(R) = \mathrm{Hom}_R(\mathbb{Q}_p/\mathbb{Z}_p, G) = \varprojlim_n G[p^n]$.

Let $G, H$ be two $p$-div. grps over $R$.

Let us denote by $\mathcal{A}_{G[p^n]}$ the coordinate ring of $G[p^n]$.

$TG$ is represented by $\mathcal{A}_G = \varinjlim_n \mathcal{A}_{G[p^n]}$ a Hopf algebra.

Let $\mathcal{A}_G^1 = \varprojlim_n \mathcal{A}_{G[p^n]}$ with maps given by immersions $G_n \hookrightarrow G_{n+1}$.

Let $G^\vee$ be the Cartier dual of $G$. We have

$$\mathcal{A}_{G^\vee} = \varinjlim_n \mathrm{Hom}_R(\mathcal{A}_{G[p^n]}, R) = \mathrm{Hom}_{R,cont}(\mathcal{A}_G^1, R).$$

(continuous $R$-mod hom.)

Let $S = \mathcal{A}_G \otimes \mathcal{A}_{H^\vee} = \mathrm{Hom}_{R,\mathrm{cont}}(\mathcal{A}_H', \mathcal{A}_G)$.
Then $S$ is an f-semiperfect ring which represents $TG \times TH^\vee$, hence there are universal morphisms

$$\mathbb{Q}_p/\mathbb{Z}_p \to G \quad , \quad \mathbb{Q}_p/\mathbb{Z}_p \to H^\vee \text{ over } S$$
$$\Leftrightarrow$$
$$H \to \mu_{p^\infty} \text{ over } S$$

hence on the level of Dieudonné modules:

$$M(\mathbb{Q}_p/\mathbb{Z}_p)(A_{cris}(S)) \to M(G)(A_{cris}(S)) \text{ and } M(H)(A_{cris}(S)) \to M(\mu_{p^\infty})(A_{cris}(S))$$

Compose them with $f: M(G)(A_{cris}(R)) \to M(H)(A_{cris}(R))$ to get:

$$\beta_f : M(\mathbb{Q}_p/\mathbb{Z}_p)(A_{cris}(S)) \to M(\mu_{p^\infty})(A_{cris}(S)) \text{ and so (by § 3)}$$
$$\eta_f \in \mathrm{Hom}_S(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty})[\tfrac{1}{p}] . \text{ Then one proves:}$$

<u>prop.</u> If $\eta_f = 0$ then $f$ is p-torsion

It is ~~now~~ enough to construct a map $\mathrm{Hom}_R(M(G), M(H))[\frac{1}{p}] \to \mathrm{Hom}_R(G,H)[\frac{1}{p}]$ which will be injective by the above proposition.

Observe that $\eta_f$ is equivalent to a family of elts $r_{f,n} : \mathcal{A}_H' \to \mathcal{A}_G$ ~~because~~ as $\mathrm{Hom}_S(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty})[\frac{1}{p}]$ is equiv. to a family of elts $s_n \in S$ $(n \in \mathbb{Z})$ s.t. $s_{n+1}^p = s_n$ and $s_n = 1$ for $n$ suff. negative

One proves that $r_{f,n} : \mathcal{A}_H' \to \mathcal{A}_G$ are morphisms of Hopf algebras. (~~ex~~ checking it by hand) and hence we get that

$$r_{f,n} \in \mathrm{Hom}_{\substack{R,\mathrm{cont} \\ R\text{-Hopf}}}(\mathcal{A}_H', \mathcal{A}_G) = \varprojlim_m \mathrm{Hom}_{R\text{-Hopf}}(\mathcal{A}_{H[p^m]}, \mathcal{A}_{G[p^m]}) = \varinjlim_m \underset{\substack{\uparrow \\ \text{in the cat.} \\ \text{of group schemes } /R}}{\mathrm{Hom}}(G[p^m], H[p^m])$$

Compatibilities between different $r_{f,n}$ as above combine to give a morphism $\Psi : G \to H$ of p-divisible groups. hence we get a map $\mathrm{Hom}_R(M(G), M(H))[\frac{1}{p}] \to \mathrm{Hom}_R(G,H)[\frac{1}{p}]$ which is an identity when composed with $\mathrm{Hom}_R(G,H)[\frac{1}{p}] \to \mathrm{Hom}_R(M(G), M(H))[\frac{1}{p}]$.

This finishes the proof of the Main Result.

$\square$