GDT W. Niziol "p-adic comparison thm." Part I

[5.1 in Scholze's p.]

Main theorem: $K/\mathbb{Q}_p$ - alg. closed, complete field, $K^+ \subset K$ open bounded

$\boxed{5.1}$ $X \to \mathrm{Spa}(K, K^+)$ proper, smooth adic space; $\mathbb{L}$ $\mathbb{F}_p$-loc. system on $X_{\acute{e}t}$

① $\forall_{i \geq 0}$ $H^i(X_{\acute{e}t}, \mathbb{L})$ finite dim $\mathbb{F}_p$-vec. space, which is $=0$ for $i > 2\dim X$.

② $\exists_{\text{almost iso.}}$ (of K^+-modules) $H^i(X_{\acute{e}t}, \mathbb{L} \otimes \mathcal{O}_X^{+a}/p) \overset{\text{alm. iso.}}{\cong} H^i(X_{\acute{e}t}, \mathbb{L}) \otimes K^{+a}/p$

Cor. (relative) $f: X \to Y$ proper, smooth, X, Y loc. noetherian adic spaces $/\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$, $\mathbb{L}$ $\mathbb{F}_p$-loc. system $/X_{\acute{e}t}$. For $i \geq 0$:

$\exists_{\text{iso.}}$ almost (of $\mathcal{O}_Y^+$-modules) $(R^i f_{\acute{e}t,*} \mathbb{L}) \otimes \mathcal{O}_Y^{+a}/p \cong R^i f_{\acute{e}t,*}(\mathbb{L} \otimes \mathcal{O}_X^{+}/p)$

proof: Take a geom. point $\mathrm{Spa}(L, L^+) \to Y$ base-change and main thm.

History of de Rham comparison. ① Faltings - by almost étale method (for constant coefficients) D

Tsuzuki - did it for smooth sheaves. (unpublished) → problems with topology?

F. and T. - algebraic setting

② Scholze - proof for proper, smooth spaces and de Rham, smooth sheaves. (no Poincaré duality needed)

Proof of thm 5.1: Assume for now

lemma 5.8: K perfectoid field of char 0, ζ_{p^n}, $n \geq 0$ (p^{th} roots of 1), X proper smooth adic space $/\mathrm{Spa}(K, \mathcal{O}_K)$, $\mathbb{L}$ $\mathbb{F}_p$-loc. system. Then:

$H^i(X_{\acute{e}t}, \mathbb{L} \otimes \mathcal{O}_X^+/p)$ are f.g. $\mathcal{O}_K$-modules and almost zero for $i > 2\dim X$

[we'll prove it later]

Step 1: pass from (K, K^+) to $(K, \mathcal{O}_K)$; $X' = X \times_{\mathrm{Spa}(K,K^+)} \mathrm{Spa}(K, \mathcal{O}_K) \underset{\text{open}}{\subset} X$

Claim: $H^i(X_{\acute{e}t}, \mathbb{L} \otimes \mathcal{O}_X^+/p) \underset{\text{iso.}}{\overset{\text{K-almost}}{\cong}} H^i(X'_{\acute{e}t}, \mathbb{L} \otimes \mathcal{O}_X^+/p)$

proof of claim: Take a hypercovering $\mathcal{U} \to X$ s.t. $\mathcal{U}_m$ affine perfectoid hence $\mathcal{U} \times_X X' \to X'$. Then

It is: (*) $H^i(\mathcal{U}_m, \mathbb{L} \otimes \mathcal{O}_{\mathcal{U}_m}^+/p) \overset{\text{a. iso}}{\cong} H^i(\mathcal{U}_m \times_X X', \mathbb{L} \otimes \mathcal{O}_{\mathcal{U}_m}^+/p)$: it is alm. 0 for $i > 0$ and alm. iso. for $i = 0$.

4.9. ⇝ 4.11. i.e.:

(a) $i > 0$ $H^i(\mathcal{U}, \mathbb{L} \otimes \mathcal{O}_{\mathcal{U}}^+/p) = 0$

(b) $i = 0$ $H^0(\mathcal{U}, \mathbb{L} \otimes \mathcal{O}_{\mathcal{U}}^+/p) = M(\mathcal{U})$ a fin. gen. projective R^{+a}/p-module

(c) $\mathcal{U}' \to \mathcal{U}$ ⇝ $M(\mathcal{U}') = M(\mathcal{U}) \otimes_{R^{+a}/p} R'^{+}/p$

Step 2: claim there exists an Artin-Schreier sequence (on $X_{proét}$):

$$0 \to \mathbb{L} \to \mathbb{L} \otimes \hat{\mathcal{O}}_{X^\flat} \to \mathbb{L} \otimes \hat{\mathcal{O}}_{X^\flat} \to 0$$

here: $\hat{\mathcal{O}}_{X^\flat} := \varprojlim \mathcal{O}_X^+/p$. Enough to prove surjectivity; can assume $U \in X_{proét}$ affinoid perfectoid and $\mathbb{L}_U$ is trivial.

need: to be able to realise étale covering as a tilt of étale covering.

have it: $\hat{U}^\flat_{fét} \cong \hat{U}_{fét}$ + descend it down

There exists a long exact sequence:

$$\dots \to H^i(X_{proét}, \mathbb{L}) \to H^i(X_{proét}, \mathbb{L}\otimes\hat{\mathcal{O}}_{X^\flat}) \to H^i(X_{proét}, \mathbb{L}\otimes\hat{\mathcal{O}}_{X^\flat}) \to \dots$$

(**) claim: $H^i(X_{proét}, \mathbb{L}\otimes\hat{\mathcal{O}}_{X^\flat}) \simeq (K^\flat)^r$ comp. with Frobenius

Assume it, then we would have: $\dots \xrightarrow{0} (\mathbb{F}_p)^r \to (K^\flat)^r \xrightarrow{x \mapsto x^p} (K^\flat)^r \xrightarrow{0} \dots$

hence $H^i(X_{proét}, \mathbb{L}) \cong \mathbb{F}_p^r$.

and then we get that $H^i(X_{et}, \mathbb{L}\otimes\mathcal{O}_X^{+a}/p) \underset{iso.}{\overset{alm.}{\cong}} H^i(X_{et}, \mathbb{L}) \otimes K^{+a}/p$

by tilting $\wr$ $\qquad\qquad\qquad\qquad \wr$

$$(K^{+a}/p)^r \cong (\mathbb{F}_p^r)\otimes K^{+a}/p.$$

proof of (**): mod π $\rightsquigarrow$ mod π^n $\rightsquigarrow$ $\varprojlim$ where $\pi \leftrightarrow p = \pi^\#$.

I. Let $M_k = H^i(X_{proét}, \mathbb{L}\otimes\hat{\mathcal{O}}_X^+/\pi^k)$. By devissage + Frobenius $\Rightarrow$
(this is fin. gen.)

$\Rightarrow$ $M_k \cong (\mathcal{O}^a_{K^\flat}/\pi^k)^r$ for some r.

II. pass to $\varprojlim$ (use lemma and $R^1\varprojlim = 0$) then invert π to get $H^i(X_{proét}, \mathbb{L}\otimes\hat{\mathcal{O}}_{X^\flat}) \cong (K^\flat)^r$ $\square$.

we have left lemma 5.8 (see above):

almost zero: start with lemma 5.6. V affinoid adic space $/\mathrm{Spa}(K, \mathcal{O}_K)$, $V \xrightarrow{étale} \mathbb{T}^n$ ("good" - composition of rational maps)

(i) $i > n \Rightarrow H^i(V_{ét}, \mathbb{L}\otimes\mathcal{O}_V^+/p) =$ almost zero.

proof: $\tilde{V} := V \times_{\mathbb{T}^n} \tilde{\mathbb{T}}^n$, $\tilde{V} \to V$ "Galois" with group $\mathbb{Z}_p^n$. S coordinate of $\tilde{V}$.

use here lemma 4.11 $\Rightarrow H^i(\tilde{V}, \mathbb{L}\otimes\mathcal{O}_V^+/p) = 0$ for $i>0$

and for $i=0$ it gives M - a fin. gen. projective S^{+a}/p -module

Use Cartan-Leray s.s. $\Rightarrow H^i(V_{proét}, \mathbb{L}\otimes\mathcal{O}_V^+/p) \underset{iso}{\overset{alm.}{\cong}} H^i_{cont}(\mathbb{Z}_p^n, M)$

coh. dim n $\rightsquigarrow$

~~[illegible]~~: $\lambda : X_{\acute{e}t} \to X_{an} \overset{(i)}{\rightsquigarrow} R^j\lambda_*(\mathbb{L} \otimes \mathcal{O}_X^+/p)$ alm. zero for $j > \dim X$.

coh. dim. $= \dim X \longrightarrow 2 \dim X$ □

f.g. $\mathcal{O}_K$-module:

(ii) $V' \subsetneq V$ rational. Then the image of $H^i(V_{\acute{e}t}, \mathbb{L} \otimes \mathcal{O}_V^+/p) \to H^i(V'_{\acute{e}t}, \mathbb{L} \otimes \mathcal{O}_{V'}^+/p)$ is almost f.g. $\mathcal{O}_K$-module.

This claims permits to conclude: choose $N = j+2$ covers $V_i^{(j+2)} \subset \dots \subset V_i^{(1)} \subset X$

For every covering $V^{(k)}$ there exists a spectral sequence

$$E_{1,(k)}^{m_1,m_2} = \bigoplus_{|J|=m_1} H^{m_2}(V^{(k)}_{J,\acute{e}t}, \mathbb{L} \otimes \mathcal{O}^+/p) \Rightarrow H^{m_1+m_2}(X_{\acute{e}t}, \mathbb{L} \otimes \mathcal{O}^+/p)$$

and get maps $E_{1,(k)} \longrightarrow E_{1,(k+1)}$

Need: Image $H^i_{cont}(\mathbb{Z}_p^n, M) \to H^i_{cont}(\mathbb{Z}_p^n, M \otimes_{S^{+a}/p} S'^{a}/p)$ is a fin. gen. $\mathcal{O}_K$-mod.

Can assume $M \sim S^{+a}/p$.

Filter $V^{(N)} = V' \subset \dots \subset V^{(k)} \subset \dots \subset V^{(1)} = V$ strict rational

Claim: $H^i_{cont}(\mathbb{Z}_p^n, S^{(j)+}/p) \longrightarrow H^i_{cont}(\mathbb{Z}_p^n, S^{(j+1)+}/p)$ image of this map is a f.g. $\mathcal{O}_K$-module.

Compute both sides: $R^+ = \mathcal{O}_K \langle T_1^{\pm 1/p^\infty}, \dots, T_n^{\pm 1/p^\infty} \rangle$

$R_m^+ = \mathcal{O}_K \langle T_1^{\pm 1/p^m}, \dots, T_n^{\pm 1/p^m} \rangle$

$U_m = \operatorname{Spa}(R_m, R_m^+)$, $V_m^{(j)} = V^{(j)} \times_X U_m$; use approximation: $S^{(j)+} \sim S_m^{(j)} \otimes_{R_m^+} R^+$

$$H^i_{cont}(\mathbb{Z}_p^n, (S_m^{(j)+} \otimes_{R_m^+} R^+)) \longrightarrow H^i_{cont}(\mathbb{Z}_p^n, S_m^{(j+1)+} \otimes_{R_m} R^+)$$

go up and down:

$$H^{i_1}((\mathbb{Z}/p^m)^n, H^{i_2}_{cont}((p^m\mathbb{Z}_p)^n, S_m^{(j)} \otimes_{R_m^+} R^+)) \Rightarrow H^{i_1+i_2}_{cont}(\mathbb{Z}_p^n, S_m^{(j)+} \otimes_{R_m^+} R^+)$$

trivial action.

get: $S_m^{(j)+}/p \otimes_{R_m^+/p} H^i_{cont}((p^m\mathbb{Z}_p)^n, R^+/p) \Rightarrow S_m^{(j+1)+} \otimes_{R_m^+} H^i_{cont}(\dots)$ what's the image of it? [Denote it IM]

(i) $S_m^{(j)+}/p \longrightarrow S_m^{(j+1)}/p$; $S_m^{(j)} \longrightarrow S_m^{(j+1)}$ compl. continuous $\Rightarrow$ mod p Im is a f.g. $\mathcal{O}_K$. [denote it A]

$$IM \Leftarrow A \otimes H^i_{cont}((p^m\mathbb{Z}_p)^n, R^+/p)$$

compute:

$$\bigoplus_{(i_1,\dots,i_n)} H^i_{cont}((p^m\mathbb{Z}_p)^n, R_m^+/p\, T_1^{i_1} \dots T_n^{i_n})$$

which is the cohomology of $\otimes (R_m^+/p \xrightarrow{\xi^{i_k} - 1} R_m^+/p)$

fix ϵ, then there are only finite number of indices which matter for us □