I. Pro-étale topology

Let X be an adic space, loc. noeth. We want to consider $V = \varprojlim V_i \to \mathcal{U} \xrightarrow{\text{ét}} X$ with $V_i \to V_j$ f. étale.

Def: Let $\mathcal{C}$ be a category. Define pro-$\mathcal{C}$ to be the category of $\varprojlim$ of objects of $\mathcal{C}$. (functors: $I \to \mathcal{C}$, I small cofiltered categories)

with morphisms: $\mathrm{Hom}(F, G) := \varprojlim_J \varinjlim_I \mathrm{Hom}(F(i), G(j))$

- $\varprojlim$ exists in pro-$\mathcal{C}$.

If X is loc. noeth. adic space (or scheme) ; $\boxed{X_{\text{profét}}}$: site with cat. pro-$X_{\text{fét}}$

$X_{\text{fét}}$: finite étale | adic spaces / schemes $/X$

coverings: $f_i : \mathcal{U}_i \to \mathcal{U}$ open maps $\bigcup |f_i(\mathcal{U}_i)| = |\mathcal{U}|$.

Assume X is connected, $\bar{x}$ geom. pt. $\Rightarrow$ $X_{\text{profét}} \cong \pi_1(X, \bar{x})$-profinite sets

Def. Γ profinite top. gp. : Γ-pfsets : cat. of profinite sets with continuous action of Γ. maps: open.

Cor. $X^{\sim}_{\text{profét}}$ has enough points

Def. 1) A morphism $\mathcal{U} \to V$ between objects in pro-$X_{\text{ét}}$ is étale (resp. finite étale) if it is induced by an étale morphism $\mathcal{U}_0 \to V_0$: $\mathcal{U} = \mathcal{U}_0 \times_{V_0} V$, $V \to V_0$ some morphism

2) $\mathcal{U} \to V$ is proétale if $\mathcal{U} = \varprojlim \mathcal{U}_i$, $\mathcal{U}_i \to V$ étale and $\forall_i \forall_{j \gg i}$ $\mathcal{U}_j \to \mathcal{U}_i$ is finite étale and surjective.

3) proétale site $X_{\text{proét}}$: cat. is the full subcat. of objects which are proétale$/X$. coverings : obvious.

It is a site!

We have to check some technicalities: · existence of fiber products

- étale / f. ét / proétale is stable under base change ; · ét / f. ét. stable under composition.

$|\mathcal{U} \times_V W| = |\mathcal{U}| \times_{|V|} |W|$

- A quasi-compact open subset of $\mathcal{U} \in \mathrm{Obj}(X_{\text{proét}})$ is also in $\mathrm{Obj}(X_{\text{proét}})$.
- existence of arbitrary finite $\varprojlim$. [we do not ask $\forall_{i,j \in I} \exists_{k \in I}\ k \geq i, j$]

Prop.: 1) Surjective f. ét. $\varprojlim$ of affinoids are quasi.c. obj. in $X_{\text{proét}}$ and form a basis for the topology, stable under fiber products.

2) $\mathcal{U} \in \mathrm{Obj}(X_{\text{proét}})$ is q.c. (resp. q.s.) iff $|\mathcal{U}|$ is.

Prop. If $x \in X$, $\exists$ morphism of topoi $Y = \mathrm{Spa}(K, K^+) \to X$:

$i_x : Y^{\sim}_{\text{proét}} \to X^{\sim}_{\text{proét}}$ s.t. $i_x^* \mathcal{F}$ = sheafification of $(V \mapsto \varinjlim_{V \to \mathcal{U}} \mathcal{F}(\mathcal{U}))$

and if $\forall_x$ $i_x^* \mathcal{F} = 0 \Rightarrow \mathcal{F} = 0$.

Comparison with étale site : $\nu : X_{\text{proét}} \to X_{\text{ét}}$

F abelian sheaf on $X_{\text{ét}}$, $\mathcal{U} = \varprojlim \mathcal{U}_i$ $(\mathcal{U}_i \in Ob(X_{\text{ét}}))$, then
$H^q(\mathcal{U}, \nu^* F) = \varinjlim H^q(\mathcal{U}_i, F)$.

proof : use Čech cohomology and Cartan-Leray spectral sequence.

Consequence : $\mathcal{F} \xrightarrow{\sim} R\nu_* \nu^* \mathcal{F}$, ($\forall i>0$ sheaf on $X_{\text{ét}}$ ass. to $V \mapsto H^i(V, \nu^* F)$ is 0) seen in $X_{\text{proét}}$

Lemma : $\mathcal{F}_i$ projective system of sheaves on $X_{\text{proét}}$, B basis for the topology s.t. $\forall_{\mathcal{U} \in B}$ $R^1 \varprojlim \mathcal{F}_i(\mathcal{U}) = 0$ and $\forall_i \forall_{q>0}$ $H^q(\mathcal{U}, F_i) = 0$ then $R^q \varprojlim F_i = 0$, hence $\forall_{\mathcal{U} \in X_{\text{proét}}} (\varprojlim \mathcal{F}_i)(\mathcal{U}) = \varprojlim \mathcal{F}_i(\mathcal{U})$ and $H^q(\mathcal{U}, \varprojlim F_i) = 0$ $\forall_{\mathcal{U} \in B}$.

II. Structure sheaves

X loc. noeth. adic space $/ \operatorname{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$; Define on $X_{\text{proét}}$:

Def. 1) $\mathcal{O}_X := \nu^* \mathcal{O}_{X_{\text{ét}}}$, $\mathcal{O}_X^+ := \nu^* \mathcal{O}^+_{X_{\text{ét}}}$

2) completed structure : $\hat{\mathcal{O}}_X^+ := \varprojlim \mathcal{O}_X^+/p^n$, $\hat{\mathcal{O}}_X := \hat{\mathcal{O}}_X^+[\frac{1}{p}]$

well-defined valuations $|f(x)|$, $x \in \mathcal{U}$, $f \in \hat{\mathcal{O}}_X$, $\mathcal{O}_X^+(\mathcal{U}) = \{ f \in \mathcal{O}_X(\mathcal{U}) \mid \forall_{x \in \mathcal{U}} |f(x)| \leq 1 \}$

$\hat{\mathcal{O}}_X^+(\mathcal{U})$ is flat over $\mathbb{Z}_p$ and p-adically complete.

Def. Let K be a perfectoid field of char 0; $X / \operatorname{Spa}(K, K^+)$. On $X_{\text{proét}}$ we have:

1) $\mathcal{U}$ is perfectoid affinoid if $\mathcal{U}$ has proétale presentation $\mathcal{U} = \varprojlim \mathcal{U}_i \to X$ with $\mathcal{U}_i = \operatorname{Spa}(R_i, R_i^+)$ s.t. letting $R^+ = \widehat{\varinjlim R_i^+}$, $R = R^+[\frac{1}{p}]$, then (R, R^+) is a perfectoid K-algebra.

2) $\mathcal{U}$ is perfectoid if it can be covered by perfectoid affinoids.

Then we can define $\hat{\mathcal{U}} = \operatorname{Spa}(R, R^+)$, and glue this construction.

Rem.: If $\mathcal{U}$ is perfectoid, $V \to \mathcal{U}$ proétale then V is perfectoid.

Prop. If X is smooth adic space $/ \operatorname{Spa}(K, K^+)$, K perf. field, then the set of $\mathcal{U} \in X_{\text{proét}}$ which are perfectoid forms a basis for the topology.

If $\mathcal{U}$ is a perfectoid affinoid in $X_{\text{proét}}$, then $\forall_{b \in K^+ \setminus \{0\}}$ $(\mathcal{O}_X^+(\mathcal{U})/b)^a = (\mathcal{O}_X^+/b)(\mathcal{U})^a$

$\hat{\mathcal{U}} = \operatorname{Spa}(R, R^+) \rightsquigarrow \hat{\mathcal{O}}_X^{(+)}(\mathcal{U}) = R^{(+)}$.

$\hat{\mathcal{O}}_X^+(\mathcal{U})$ is the p-adic completion of $\mathcal{O}_X^+(\mathcal{U})$; $H^q(\mathcal{U}, \hat{\mathcal{O}}_X^+)^a = 0$ for $q > 0$.

III. Period sheaves

using the fact that perfectoid eff. form a basis

Def. Let X be a loc. noeth. adic space $/ \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. We have sheaves on $X_{\text{proét}}$:

1) $\mathbb{A}_{\inf} = W(\hat{\mathcal{O}}^+_{X^\flat})$ $\left(= \varprojlim W_n(\hat{\mathcal{O}}^+_{X^\flat}) \right)$, $\mathbb{B}_{\inf} = \mathbb{A}_{\inf}[\frac{1}{p}]$ still have $\theta : \mathbb{A}_{\inf} \to \hat{\mathcal{O}}^+_X$, $\theta : \mathbb{B}_{\inf} \to \hat{\mathcal{O}}_X$

2) $\mathbb{B}^+_{dR} = \varprojlim \mathbb{B}_{\inf} / (\ker \theta)^n$ $\quad \mathbb{B}_{dR} = \mathbb{B}^+_{dR}[\frac{1}{t}]$

(locally, there exists a t, unique up to a unit, which generates $\mathrm{Fil}^1(\mathbb{B}^+_{dR})$)

$\mathrm{Fil}^i \mathbb{B}^+_{dR} = (\ker\theta)^i \mathbb{B}^+_{dR}$.

Rem.: If (R, R^+) is perfectoid K-alg., $\mathbb{A}_{\inf}(R, R^+) = W(R^{\flat,+})$ and similarly we have all the other rings.

Lemma: $\exists \xi \in \mathbb{A}_{\inf}(R, R^+)$ which generates $\ker\theta$. ξ is not a zero-divisor.

Proof: $\pi \in K^\flat$ s.t. $\pi^\#/p \in (K^+)^\times$. Define $\xi = [\pi] - \sum_{i \geq 1} p^i [x_i]$ where x_i are s.t. $\xi \in \ker\theta$ $\square$

Rem: $gr^i(\mathbb{B}_{dR}(R, R^+)) \cong \xi^i R$.

Thm. Let $U \in \mathrm{Ob}(X_{\text{proét}})$ eff. perfectoid, with $\hat{U} = \mathrm{Spa}(R, R^+)$

1) $\exists$ a can. isom. $\mathbb{A}_{\inf}(U) \cong \mathbb{A}_{\inf}(R, R^+)$ and similarly for other rings.

2) $H^q(U, F)^a = 0$ where $F \in \{\mathbb{A}_{\inf}, \dots, \mathbb{B}_{dR}\}$

3) In $\mathbb{B}_{dR}(U)$, $[\pi]$ is invertible.

For $\mathbb{B}^+_{dR}$ we have $0 \to \mathbb{B}_{\inf} \xrightarrow{\times \xi^i} \mathbb{B}_{\inf} \to \mathbb{B}_{\inf}/(\ker\theta)^i \to 0$ is exact and stays exact for section on U as $H^1(U, \mathbb{B}_{\inf})^a = 0$.

$gr^i \mathbb{B}_{dR} \cong \hat{\mathcal{O}}_X(i)$, $\left(\mathbb{Z}_p(1) = \varprojlim \mu_{p^n} \right)$

Def. Let k be a discretely valued complete ext. of $\mathbb{Q}_p$ with perfect res. field K.
Let X be a noeth. adic space $/ \mathrm{Spa}(k, \mathcal{O}_k)$. We have on $X_{\text{proét}}$:

1) $\Omega^1_X = \nu^* \Omega^1_{X_{\text{ét}}}$ $(\rightsquigarrow \Omega^k)$

2) $\mathcal{O}\mathbb{B}_{\inf} = \mathcal{O}_X \otimes_{W(k)} \mathbb{B}_{\inf}$ (constant sheaf), $\mathcal{O}\mathbb{B}^+_{dR} = \varprojlim \dots$, $\mathcal{O}\mathbb{B}_{dR} = \mathcal{O}\mathbb{B}^+_{dR}[\frac{1}{t}]$.

$\mathcal{O}\mathbb{B}_{inf}$ has a unique extension of the connection $\nabla : \mathcal{O}_X \to \Omega^1_X$:
$\nabla : \mathcal{O}\mathbb{B}_{inf} \to \mathbb{B}_{inf} \otimes \Omega^1_X$ $\rightsquigarrow$ extension to $\mathcal{O}\mathbb{B}_{dR}$.

Define $\mathbb{T}^n := \mathrm{Spa}(k\langle T_1^{\pm 1}, \dots, T_n^{\pm 1}\rangle, k^+\langle T_1^{\pm 1}, \dots, T_n^{\pm 1}\rangle)$ [torus]

We have an isomorphism $\mathcal{O}\mathbb{B}_{dR}^+ \cong \mathbb{B}_{dR}^+[[x_1, \dots, x_n]]$ $(n = \dim X)$
mapping X_i to $T_i \otimes 1 - 1 \otimes [T_i^\flat]$

Poincaré lemma: (for any smooth X of dimension n)
$$0 \to \mathbb{B}_{dR}^+ \to \mathcal{O}\mathbb{B}_{dR}^+ \xrightarrow{\nabla} \mathcal{O}\mathbb{B}_{dR}^+ \otimes \Omega^1_X \xrightarrow{\nabla} \dots \to \mathcal{O}\mathbb{B}_{dR}^+ \otimes \Omega^n_X \to 0$$
is an exact sequence of sheaves on $X_{proét}$. We have Griffiths transversality :
$$\nabla(\mathrm{Fil}^i \mathcal{O}\mathbb{B}_{dR}^+) \subset \mathrm{Fil}^{i-1}(\mathcal{O}\mathbb{B}_{dR}^+) \otimes \Omega^1_X$$
and letting Ω^i_X have degree 1, the sequence is strict exact.
In particular, gr^i is an exact sequence, for example for $i = 1$:
$$0 \to \hat{\mathbb{Q}}_X(1) \to \mathrm{gr}^1 \mathcal{O}\mathbb{B}_{dR}^+ \to \hat{\mathcal{O}}_X \otimes \Omega^1_X \to 0 \text{ is exact.}$$
for the torus : $\mathrm{gr}^i \mathcal{O}\mathbb{B}_{dR} \cong \xi^i \hat{\mathcal{O}}_X [\frac{x_1}{\xi}, \dots, \frac{x_n}{\xi}]$

Prop. $X = \mathrm{Spa}(R, R^+)$ affinoid adic space $/\mathrm{Spa}(k, \mathcal{O}_k)$ with an étale map to $\mathbb{T}^n$ which factors as a composition of fét maps and rational embeddings.
1) If K/k perfectoid and $\mu_{p^\infty} \subset K$, then $H^q(X_K, \mathrm{gr}^0 \mathcal{O}\mathbb{B}_{dR}) = 0$ for $q > 0$.
and $H^0(X_K, \mathrm{gr}^0 \mathcal{O}\mathbb{B}_{dR}) = R \hat{\otimes}_k K$.
2) $H^q(X, \mathrm{gr}^i \mathcal{O}\mathbb{B}_{dR}) = 0$ unless $i = 0$ and $q = 0$ or 1 : $i=0, q=0$: R; $i=0, q=1$: $R\log(\chi)$
where $\chi : \mathrm{Gal}(\bar{k}/k) \to \mathbb{Z}_p^\times$ cyclotomic character.

proof: reduce to the torus : $\mathrm{Spa}(K\langle T_1^{\pm 1/p^\infty}, \dots, T_n^{\pm 1/p^\infty}\rangle, K^+\langle T_1^{\pm 1/p^\infty}, \dots, T_n^{\pm 1/p^\infty}\rangle)$
which has Galois group $\mathbb{Z}_p^n$. Plus, use the Cartan-Leray spectral sequence.

Cor (of Poincaré lemma) X smooth adic space over $\mathrm{Spa}(K, \mathcal{O}_K)$. Then $\nu_* \mathcal{O}\mathbb{B}_{dR} = \mathcal{O}_{X_{ét}}$
and $\nu_* \hat{\mathcal{O}}_X = \mathcal{O}_{X_{ét}}$, $\nu_* \hat{\mathcal{O}}_X(n) = 0$ if $n \geq 1$.
We have also $R^1\nu_* \hat{\mathbb{Q}}_X(1) \cong \Omega^1_{X_{ét}}$.

proof: ~~proof~~ use Poincaré's lemma and compute.