FROM PERFECTOID SPACES VIA THE FARGUES-FONTAINE CURVE TO RAPOPORT-ZINK SPACES AT INFINITY

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This text is based on the lecture I have given on the 1st October 2012 as an introduction to the seminar in Paris 6 on adic spaces, perfectoid spaces and the p-adic Hodge theory. As such, it contains many imprecisions and simplifications.

1. Adic spaces and perfectoid spaces

Let k be a nonarchimedean field, by which we mean a topological field with topology induced by a rank 1 valuation.

Definition 1.1. *A Tate k-algebra is a topological k-algebra R for which there exists a subring $R_0 \subset R$ such that aR_0 (for $a \in k^\times$) forms a basis of open neighbourhoods of 0. A subset $M \subset R$ is called bounded if $M \subset aR_0$ for some $a \in k^\times$. An element $x \in R$ is called powerbounded if $\{x^n \mid n \geq 0\}$ is bounded. Let $R^\circ \subset R$ denote the set of powerbounded elements.*

Definition 1.2. *An affinoid k-algebra is a pair (R, R^+) consisting of a Tate k-algebra R and an open, integrally closed subring $R^+ \subset R^\circ$. It is said to be of topologically finite type if R is a quotient of $k\langle T_1, ..., T_n\rangle = \{\sum x_{i_1 \ldots i_n} T_1^{i_1} \ldots T_n^{i_n} \mid x_{i_1 \ldots i_n} \to 0\}$ for some n and $R^+ = R^\circ$.*

We can now define for an affinoid k-algebra (R, R^+) an affinoid adic space

$$X = \mathrm{Spa}(R, R^+) = \{|.| : R \to \Gamma \cup \{0\} \text{ continuous valuation } |\forall_{f \in R^+} |f| \leq 1\}/_\equiv$$

as the set of equivalence classes of certain continuous valuations on R. Here Γ is some totally ordered abelian group written multiplicatively (Γ changes with $|.|$) and by continuous we mean that for each $\gamma \in \Gamma$ the set $\{x \in R \mid |x| < \gamma\} \subset R$ is open. For each $x \in X$ we write $f \mapsto |f(x)|$ for the corresponding valuation on R, that is $x(f)$.

We equip X with the topology which has the open subsets

$$U\left(\frac{f_1, ..., f_n}{g}\right) = \{x \in X \mid \forall_i |f_i(x)| \leq g(x)\}$$

(called rational subsets) where $f_1, ..., f_n \in R$ generate R as an ideal and $g \in R$. One then is able to define presheaves $\mathcal{O}_X$ and $\mathcal{O}_X^+$ on X by giving them on rational subsets, by $\mathcal{O}_X(U(\frac{f_1,...,f_n}{g})) = R\left\langle \frac{f_1}{g}, ..., \frac{f_n}{g} \right\rangle$ = completion of $R[\frac{f_1}{g}, ..., \frac{f_n}{g}]$ and $\mathcal{O}_X^+(U(\frac{f_1,...,f_n}{g}))$ = completion of integral closure of $R^+[\frac{f_1}{g}, ..., \frac{f_n}{g}]$ in $R[\frac{f_1}{g}, ..., \frac{f_n}{g}]$. We remark that in general $\mathcal{O}_X$ and $\mathcal{O}_X^+$ need not to be sheaves (but there are always for perfectoid spaces).

We are led to the following definition

Definition 1.3. *An adic space is a locally ringed space locally isomorphic to an affinoid adic space. In fact it is a quadruple $(X, \mathcal{O}_X, \mathcal{O}_X^+, |.(x)| \forall_{x \in X})$ which consists of a spaces, two presheaves and a valuation $|.(x)|$ on each stalk $\mathcal{O}_{X,x}$ (see* [Hu1] *or* [Sch1]*).*

Now we come to perfectoid spaces which are a particular class of adic spaces

Definition 1.4. *A perfectoid field is a complete non-archimedean field K of residue characteristic $p > 0$ whose associated rank 1 valuation is non-discrete and such that the Frobenius is surjective on K°/p*

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Remark 1.5. *Standard examples are the p-adic completion of* $\mathbb{Q}_p(p^{1/p^\infty})$ *and the t-adic completion of* $\mathbb{F}_p((t))(t^{1/p^\infty})$.

There is a process of passic from the category of all perfectoid fields to the category of perfectoid fields of characteristic $p > 0$, which is called tilting. Choose $\varpi \in K^\times$ such that $|p| \leq |\varpi| < 1$ and consider $\varprojlim_\Phi K^\circ/\varpi$ where Φ is the Frobenius $x \mapsto x^p$. This gives a perfect ring of characteristic p, equipped with the inverse limit topology, where we put on each K°/ϖ discrete topology. One proves

Lemma 1.6. *(i) There exists a multiplicative homeomorphism* $\varprojlim_{x\mapsto x^p} K^\circ \simeq \varprojlim_\Phi K^\circ/\varpi$ *given by projection. Hence we have a map* $\varprojlim_\Phi K^\circ/\varpi \to K^\circ$ *which we denote* $x \mapsto x^\#$.

(ii) There exists an element $\varpi^\flat \in \varprojlim_\Phi K^\circ/\varpi$ *with* $|(\varpi^\natural)^\#| = |\varpi|$. *Define the tilt of* K *as*

$$K^\flat = (\varprojlim_\Phi K^\circ/\varpi)[(\varpi^\flat)^{-1}]$$

(iii) There exists a multiplicative homeomorphism $K^\flat \simeq \varprojlim_{x\mapsto x^p} K$, *in particular there exists a map* $K^\flat \to K$ *written* $x \mapsto x^\#$. $K^\flat$ *is a perfectoid field and* $K^{\flat\circ} = \varprojlim_{x\mapsto x^p} K^\circ \simeq \varprojlim_\Phi K^\circ/\varpi$. *One can define a valuation on* $K^\flat$ *by* $|x|_{K^\flat} = |x^\#|_K$

Fix now a perfectoid field K.

Definition 1.7. *A perfectoid* K*-algebra is a Banach* K*-algebra* R *such that the subset* $R^\circ \subset R$ *of powerbounded elements is open and bounded, and the Frobenius morphism is* $\Phi : R^\circ/\varpi \to R^\circ/\varpi$ *is surjective. Morphisms between perfectoid* K*-algebras are the continuous morphisms of* K*-algebras.*

Similarly as for perfectoid fields, we can define the tild of R. While the real definition passes by almost mathematics (see [GR]), we will content ourselves with putting $R^\flat = \varprojlim_{x\mapsto x^p} R$, which comes out as a result of a definition given in [Sch1]. We still have a map $x \mapsto x^\# : R^\flat \to R$.

Remark 1.8. *A standard example of a perfectoid* K*-algebra is* $R = K\left\langle T_1^{1/p^\infty}, ..., T_n^{1/p^\infty}\right\rangle = K^\circ[\widehat{T_1^{1/p^\infty}, ..., T_n^{1/p^\infty}}][1/\varpi]$. *Its tilt is* $R^\flat = K^\flat\left\langle T_1^{1/p^\infty}, ..., T_n^{1/p^\infty}\right\rangle$.

Definition 1.9. *A perfectoid affinoid* K*-algebra is an affinoid* K*-algebra* (R, R^+) *such that* R *is a perfectoid* K*-algebra.*

We still have tilting for spaces:

$$X = \mathrm{Spa}(R, R^+) \mapsto X^\flat = \mathrm{Spa}(R^\flat, R^{\flat +})$$

Theorem 1.10. *(i) We have a homeomorphism* $X \simeq X^\flat$ *given by* $x \mapsto x^\flat$, *where* $x^\flat$ *is such that* $|f(x^\flat)| = |f^\#(x)|$. *This homeomorphism identifies rational subsets.*

(ii)For each $U \subset X$ *rational subset,* $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ *is perfectoid with the tilt* $(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat))$.

(iii) Presheaves $\mathcal{O}_X$ *and* $\mathcal{O}_X^+$ *are sheaves.*

Definition 1.11. *A perfectoid space is an adic space over* K *that is locally isomorphic to an affinoid perfectoid.*

A process of tilting gluess well and hence we have tilting for general perfectoid spaces: $X \mapsto X^\flat$. We come to the main property of perfectoid spaces. Namely, one can define a notion of étale morphism between perfectoid spaces, thus the étale site X_{et}. Then:

Theorem 1.12. *We have an equivalence of étale topoi of tilts*

$$X_{et} \simeq X_{et}^\flat$$

This for example largely generalizes the theorem of Fontaine-Winterberger about the isomorphism of absolute Galois groups:

$$\mathrm{Gal}(\overline{\mathbb{Q}_p(p^{1/p^\infty})}/\mathbb{Q}_p(p^{1/p^\infty})) \simeq \mathrm{Gal}(\overline{\mathbb{F}_p((t))}/\mathbb{F}_p((t)))$$

Let us remark, that the reason of introducing perfectoid spaces and its first application is actually the weight-monodromy conjecture of Deligne, which we do not discuss. See [Sch1].

2. p-adic Hodge theory

Perfectoid spaces suit well the purpose of proving a comparison theorem between p-adic étale and de Rham cohomologies. This is because they form a basis for the topology in the pro-étale site. Scholze proves (see [Sch2])

Theorem 2.1. *Let k be a discretely valued complete non-archimedean extension of $\mathbb{Q}_p$ with perfect residue field κ, and let X be a proper smooth rigid-analytic variety (i.e. a locally noetherian adic space) over k. For a filtered module with integrable connection $(\mathcal{E}, \nabla, Fil^\bullet)$ on X_{et} with associated $\mathbb{B}_{dR}^+$-local system $\mathbb{M}$, we have a $\mathrm{Gal}(\bar{k}/k)$-equivariant isomorphism*

$$H^i_{dR}(X_{\bar{k}}, \mathcal{E}) \otimes_k B_{dR} \simeq H^i(X_{\bar{k}}, \mathbb{M}) \otimes_{B_{dR}^+} B_{dR}$$

We have to explain the terminology used in the formulation above. First of all, $\mathbb{B}_{dR}^+$ and also $\mathcal{O}\mathbb{B}_{dR}^+$ are sheafified Fontaine's rings living on the pro-étale site of X. We do not introduce them here, refering to [Sch2] for a precise definition. By $\mathbb{M}$ being associated to $\mathcal{E}$ we mean that there exists an isomorphism

$$\mathcal{E}_{\mathcal{O}_X} \mathcal{O}\mathbb{B}_{dR}^+ \simeq \mathbb{M} \otimes_{\mathbb{B}_{dR}^+} \mathcal{O}\mathbb{B}_{dR}^+$$

During the seminar we will cover the proof of the above theorem and let us now give a simple sketch of it. The proof basically follows the proof of Andreatta-Iovita ([AI]) of B_{cris}-conjecture for p-adic formal schemes, but by introducing the pro-étale site and using perfectoid spaces Scholze is able to simplify it a lot. Let us review this notion.

Let X be a locally noetherian adic space (or scheme). Let X_{et} be the category of spaces Y étale over X. Let $pro - X_{et}$ be the category of pro-objects in X_{et}. To $U = \varprojlim_i U_i \in pro - X_{et}$ we associate a topological space $|U| = \varprojlim_i |U_i|$. A morphism $U \to V$ in $pro - X_{et}$ is étale (resp. finite étale) if $U = U_0 \times_{V_0} V$ for some morphism $V \to V_0$ and $U_0 \to V_0$ is étale (resp. finite étale) in X_{et}. A morphism $U \to V$ in $pro - X_{et}$ is pro-étale if it can be written as a cofiltered inverse limit $U = \varprojlim_i U_i$ of objects $U_i \to V$ étale over V such that $U_i \to U_j$ is finite étale and surjective for large $i > j$ (remark that $U_i \in pro - X_{et}$).

Definition 2.2. *The pro-étale site X_{proet} has as an underlying category the full subcategory of $pro - X_{et}$ of objects that are pro-étale over X. A covering in X_{proet} is given by a familly of pro-étale morphisms $\{f_i : U_i \to U\}$ such that $|U| = \bigcup_i f_i(|U_i|)$.*

One introduces a notion of an affinoid perfectoid in X_{proet} which is basically an inverse limit of $\mathrm{Spa}(R_i, R_i^+)$ with R_i perfectoid. Those glue together to a notion of a perfectoid space in X_{proet}. A standard example (and in fact, crucial in applications) of an affinoid perfectoid is

$$\tilde{\mathbb{T}}^n = \varprojlim_m \mathrm{Spa}(K \left\langle T_1^{\pm 1/p^m}, ..., T_n^{\pm 1/p^m} \right\rangle, K^+ \left\langle T_1^{\pm 1/p^m}, ..., T_n^{\pm 1/p^m} \right\rangle)$$

Strength of a pro-étale site lies in the following theorem (see Proposition 4.8 in [Sch2])

Theorem 2.3. *Let X be a locally noetherian adic space over $\mathrm{Spa}(K, K^+)$. Then the set of $U \in X_{proet}$ which are affinoid perfectoid form a basis for the topology.*

Having this in hand, one is able to define and verify basic properties of sheafified Fontaine's rings by checking them on affinoid perfectoids. Then, to approach the comparison theorem, one establishes a version of Poincare's lemma

$$0 \to \mathbb{B}_{dR}^+ \to \mathcal{O}\mathbb{B}_{dR}^+ \otimes_{\mathcal{O}_X} \Omega_X^1 \to ... \to \mathcal{O}\mathbb{B}_{dR}^+ \otimes_{\mathcal{O}_X} \Omega_X^n \to 0$$

where n is the dimension of X. This permits to follow the strategy of proof of the classical comparison theorem (between Betti and de Rham cohomology). Introduce a de Rham complex associated to our sheaf $\mathcal{E}$

$$DR(\mathcal{E}) = (0 \to \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_X^1 \to ...)$$

and consider a morphism which we get from the Poincare's lemma:

$$R\Gamma(X_{\bar{k}}, DR(\mathcal{E})) \otimes_k B_{dR} \to R\Gamma(X_{\bar{k}}, DR(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{B}_{dR})$$

By looking at the graded pieces of it, one can show that the above map is a quasi-isomorphism, thus finishing the proof.

3. The Fargues-Fontaine curve and Rapoport-Zink spaces at infinity

The last part of the seminar will concentrate on applications of perfectoid spaces to p-divisible groups. Especially, Scholze and Weinstein in [SW] (and Weinstein in [We] for Lubin-Tate tower in positive characteristic) show that the Rapoport-Zink spaces at infinity (i.e. an inverse limit over all finite levels) are perfectoid spaces. This leads to further applications (for example Faltings isomorphism between two Rapoport-Zink towers and a theorem on the image of period morphism).

We consider a p-divisible group G_0 over the residue field of K of dimension 1 and height n. The [SW] introduces the Lubin-Tate perfectoid $\mathcal{M}_{G_0}$ (actually they do it in much more general setting) which is a perfectoid space classifying deformations of G_0 with a level structure. Morphisms of $\mathrm{Spa}(R, R^+)$ into $\mathcal{M}_{G_0}$ are in functorial bijection with isogeny classes of deformations of G_0 with level structure over R^+. If we let F be a finite extension of $\mathbb{Q}_p$ then an $\mathcal{O}_F$-level structure is an isomorphism

$$\alpha : F^n \to V(G)(R) = \varprojlim_n G_n(R) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

One gets a particularly simple form of $\mathcal{M}_{G_0}$, namely we have a fibre product in the category of adic spaces

$$\begin{array}{ccc} \mathcal{M}_{G_0} & \longrightarrow & \widetilde{G_0}^n \\ \downarrow & & \downarrow \\ V(\wedge G) & \longrightarrow & \widetilde{\wedge G_0} \end{array}$$

where $\widetilde{G_0}$ is a universal cover of G_0 defined by

$$\widetilde{G_0}(R) = \varprojlim_{x \mapsto px} G_0(R)$$

We have denoted by $\wedge G_0$ the determinant of G_0 and G denotes any lift of G_0 to R^+.

Scholze and Weinstein are able to describe $Spa(R, R^+)$-points of a general Rapoport-Zink space at infinity by classifying firstly p-divisible groups over $\mathcal{O}_C$, where C is an algebraically closed extension of $\mathbb{Q}_p$. They do it by showing that the Dieudonné module functor is fully faithful. This point is connected to the Fargues-Fontaine curve ([FF]) as the p-divisible groups might be naturally considered as vector bundles over the Fargues-Fontaine curve (this allows to prove one technical lemma in [SW]). On the other hand, there is a conjecture (see [Fa] for details), which gives a different description of p-divisible groups over $\mathcal{O}_C$, namely in terms of Kisin modules over the Fargues-Fontaine curve. It would be interesting to investigate what these two points of view have in common.

Let us finish, by recalling the definition of the Fargues-Fontaine curve. Let (R, R^+) be an affinoid perfectoid K-algebra with a norm $|.|$ on R. We define a family of norms $|.|_r$ $(r > 0)$ on Witt vectors $W(R^+)[\frac{1}{p}]$ by

$$|\sum_{n>>-\infty} [x_n]p^n|_r = sup\{|x_n|p^{-rn}\}_n$$

Let $A_r \subset W(R^+)[\frac{1}{p}]$ be the subring of elements x such that $|x|_r \leq 1$. Put $B_r = A_r[\frac{1}{p}]$ and $B(R^+) = \bigcap_{r>0} B_r$. There is an action of Frobenius ϕ on $B(R^{\flat+})$ and hence we can consider the graded algebra

$$P = \bigoplus_{d\geq 1} B(R^{\flat+})^{\phi=\pi^d}$$

where π is a uniformiser of K. We let

$$X = \mathrm{Proj}\, P$$

This X is the Fargues-Fontaine curve ([FF]). Now, for the aforementioned connection with p-divisible groups, if G_0 is a p-divisible group over the residue field of K, then let $M = M(G_0)$ be its covariant Dieudonné module. We can form a graded P-module

$$\widetilde{M} = \bigoplus_{d \geq 0} (M \otimes B(R^{\flat +}))^{\phi = \pi^d}$$

which gives rise to a vector bundle $\mathcal{M}$ on X. Hence, we have constructed a (fully faithful) functor from p-divisible groups over $\mathcal{O}_C/(p)$ to vector bundles over X. Combining this with a complete classification of vector bundles over X which was obtained in [FF], we can prove one part of [SW] (isotriviality of p-divisible groups over $\mathcal{O}_C$). This point of view is also fruitful for giving a description of $Spa(R, R^+)$-points of Rapoport-Zink spaces at infinity.

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