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Representations of $GL(n, \mathbb{Q}_p)$

$G = GL_n(\mathbb{Q}_p)$ (more generally: reductive F-group, F local field).

$R(G) = \{$smooth, admissible G-reps over $\mathbb{C}$.$\}$

$\mathcal{H}(G) = \{$compactly supported, locally constant $f: G \to \mathbb{C}\}$

— an algebra wrt <u>convolution</u> (need Haar measure to define it) assoc, non-unital alg.

<u>Equivalence of cats</u>:

$$R(G) \cong \{\text{non-deg. } \mathcal{H}(G)\text{-modules}\}$$

$$\pi \longmapsto \pi(f)v = \int_G f(x)\pi(x)v\,dx.$$
$$= \langle f, x\cdot v\rangle$$

<u>General tools</u>

1) <u>Parabolic induction</u> $P = MN$ parabolic of G, define $i_P^G: R(M) \to R(G)$

$$i_P^G(\pi) = \{f: G \to V \text{ locally constant}: f(gmn) = \delta_P(mn)^{-1/2}\pi(m^{-1})f(g)\}$$

this is <u>normalised</u> induction.

<u>Reason</u> we want i_P^G to preserve unitary reps.

2) Restriction (Jacquet) functors

$$r_P^G : R(G) \longrightarrow R(M)$$
$$(\pi, V) \longmapsto V_N = V / V.(N-1).$$

- $r_P^G(V)$ is admissible.

1st adjointness theorem

i_P^G is right adjoint to (normalised) r_P^G.

Remark $K_0^{adm}(G) := K_0(R(G))$.

On the level of K_0^{adm}, i_P^G and r_P^G only depend on M. (if you choose a different P with Levi M, composition factors might get shuffled around).

Plancherel formula

carries Fell topology

There is a unique positive Borel measure $\hat{\mu}$ on $G^\wedge$ (the unitary dual of G) s.t.

$$f(1) = \int_{\pi \in \hat{G}} \operatorname{tr} \pi(f) \, d\hat{\mu}(\pi) \qquad \forall f \in \mathcal{H}(G).$$

Thm. $\pi \in G^\wedge$ (irred unitary rep)
$\Rightarrow \pi^\infty \in R(G)$ (it is pre-unitary; has a pos. def. G-invariant herm. form)

Note:

f loc. const $\Rightarrow f \in C_c^\infty(G)^K$ some compact open K
$\Rightarrow \pi(f) V \subseteq V^K$ which is finite dimensional.

③

Defn If $\pi \in \hat{G}$ (or $\pi^\infty \in \mathcal{R}(G)$) appears in $\operatorname{Supp} \hat{\mu}$ then we call π <u>tempered</u>

π is <u>square integrable</u> if $\hat{\mu}(\pi) > 0$. ($\Leftrightarrow$ appears in $L^2(G)$)

<u>Particularly for $GL(n)$:</u>

— <u>essentially</u> tempered (or <u>essentially sq. int</u>) : <u>twists</u> of tempered (resp. sq. int) reps.

• <u>Parabolic form of the Langlands classification</u>

$i_P^G(\pi)$ where π is ess. tempered, irreducible in $\mathcal{R}(M)$, with "some dominance condition".

$$GL(n): \quad M = GL(n_1) \times \cdots \times GL(n_r)$$
$$\pi_1[a_1] \otimes \cdots \otimes \pi_r[a_r]$$

$$\pi_i[a_i] = \pi_i |\det|^{a_i} \quad a_i \in \mathbb{C}$$
π_i is tempered.

dominance: $\operatorname{Re}(a_1) > \operatorname{Re}(a_2) > \cdots > \operatorname{Re}(a_r)$.

<u>Thm</u> • each standard module has a unique irreducible quotient $\boxplus_{i=1}^{r} \pi_i[a_i]$

• every irred in $\mathcal{R}(G)$ occurs in this way.

• Two Langlands quotients are isomorphic only if the corresponding data is the same.

<u>What's special about $GL(n)$:</u>

in the Theorem, one can replace "tempered" with "square integrable", and "dominant" with "weakly dominant" (i.e. $\operatorname{Re}(a_1) \geq \operatorname{Re}(a_2) \geq \ldots$)

④ <u>Combinatorial description</u> (Bernstein-Zelevinsky) multi-segments.

<u>Defn</u> Irred $(\pi, V) \in R(G)$ is <u>cuspidal</u> if $r_P^G(\pi) = 0$ for all proper parabolic $P \subsetneq G$.

<u>Cuspidal Support</u>: if (π, V) is irreducible then π is a composition factor of $i_P^G(\rho)$, (ρ is cuspidal for M)
Then ρ, or (ρ, M) is the <u>cuspidal support</u> of π.

(eg $M = GL(n_1) \times \cdots \times GL(n_r)$, think of cuspidal support as a multiset $\{\rho_1, \cdots, \rho_r\}$)

Fix π_0, a cuspidal representation of $GL(d)$, where $d \mid n$. A <u>segment</u> is a set of the form $[a, b] := \{a, a+1, \ldots, b\}$.

Define $i^{GL(n)}_{\underbrace{GL(d) \times \cdots \times GL(d)}_{n/d}} \left(\pi_0 |\det|^a \otimes \cdots \otimes \pi_0 |\det|^b\right)$

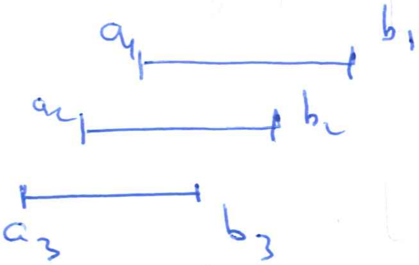
BZ: this has a <u>unique</u> quotient $\pi_0[a, b]$ which is essentially square integrable.

<u>Multisegment</u>: $[a_1, b_1], \ldots, [a_t, b_t]$ segments.
We say $[a_i, b_i]$ <u>precedes</u> $[a_j, b_j]$ if $[a_i, b_i] \cup [a_j, b_j]$ is a segment but not equal to $[a_i, b_i]$ or $[a_j, b_j]$.

Thm If $\forall i<j$ $[a_i, b_i]$ doesn't preceed $[a_j, b_j]$, then $i_P^G(\pi_0[a_1,b_1] \otimes \cdots \otimes \pi_0[a_t, b_t])$ has a unique irreducible quotient $\boxplus_{i=1}^{t} \pi_0[a_i, b_i]$

Generalised Speh representations

$[a_1, b_1], \ldots, [a_t, b_t]$ $\quad a_{i-1} = a_i - 1$. $\quad b_{i-1} = b_i - 1$.



$\boxplus_{i=1}^{t} \pi_0[a_i, b_i]$ is called a Generalised Speh rep.

(Tadić: these are the building blocks of the unitary dual of $GL(n)$).

BGG formula: $\pi = \boxplus_{i=1}^{t} \pi_0[a_i, b_i]$ Speh.

$w \in S_t$, $I_{w \cdot t} := i_{P_w}^G\left(\pi_0[a_{w(1)}, b_1] \otimes \cdots \otimes \pi_0[a_{w(t)}, b_t]\right)$

Then: $$\pi = \sum_{w \in S_t} \operatorname{sgn}(w) I_{w \cdot \pi}.$$

Particular case: $a_i = b_i = \frac{t+1}{2} - i$.

(Scholze) $\quad \pi = \boxplus_{1}^{t} \pi_0\left[\frac{t+1}{2} - i\right]$.

In $K_0^{adm}(G)$: $\boxplus_{1}^{t} \pi_0\left[\frac{t+1}{2} - i\right] + (-1)^t \pi_0\left[\frac{1-t}{2}, \frac{t-1}{2}\right]$

$=$ combination of proper induced modules

Density and trace Paley-Wiener Theorem

(Bernstein – Deligne – Kazhdan)

$$tr : \mathcal{H}(G) \longrightarrow R(G)^* = K_0(R(G))^*.$$
$$f \longmapsto (\pi \longmapsto tr(f|\pi))$$

Thm 1) (BDK) tr is surjective onto $R(G)^*_{good}$

2) (K) $\ker tr = [\mathcal{H}(G), \mathcal{H}(G)]$ the commutator subspace.

$$\Rightarrow K_0(R(G))^*_{good} \cong \mathcal{H}/[\mathcal{H},\mathcal{H}].$$

3) (Borel-Wallach) If $tr(f|\pi) = 0 \ \forall \pi$ irred tempered, then $tr(f|\pi) = 0 \ \forall \pi$ irred.

Form elliptic space $\quad i_M^G : K_0^{adm}(M) \longrightarrow K_0^{adm}(G)$

$$\overline{K(G)} = K_0^{adm}(G) \Big/ \sum_{M \neq G} i_M^G(K_0^{adm}(M)).$$

In $\overline{K(G)}$ a square integrable rep

$$= \pm \, \mathrm{Speh} \mathop{\boxplus}_{1}^{t} \pi_0\left[\frac{t+1}{2} - i\right]$$

Relation with Weil group side

Weil–Deligne group

$$W_F' = W_F \times SL_2(\mathbb{C})$$

Reps $\tau : W_F' \longrightarrow GL_n(\mathbb{C})$ have to be W_F-semisimple and $SL_2(\mathbb{C})$-algebraic.

Then $\tau = \sum_{i=1}^{r} \tau_i \otimes V(t_i)$ where

τ_i is an irred W_F-rep

and $V(t_i)$ a t_i-dim. $SL_2(\mathbb{C})$-rep.

Suppose we have Langlands correspondence for supercuspidal reps $\tau_i \longmapsto \pi_i$.

Then $\tau_i \otimes V(t_i) \longmapsto \pi_i\left[\frac{1-t_i}{2}, \frac{t_i-1}{2}\right]$

ess. sq. int.

and $\tau \longmapsto$ Langlands quotient $\boxplus_{i=1}^{r} \pi_i\left[\frac{1-t_i}{2}, \frac{t_i-1}{2}\right]$

(after arranging the segments