p-divisible groups. (Netan Dogra) ①

1. Basic defns/properties.
2. classification over finite fields.
3. ~~[illegible]~~ Deformations

1.

Defn. A p-div. gp over a scheme S is a family $(G_n)_{n\geq 0}$ of finite flat group schemes together with morphisms $G_n \xrightarrow{i_n} G_{n+1}$ s.t. G_n is $[p^n]$-torsion & i_n is an immersion. G_n is of height h if the degree of $G_n \to S$ is p^{nh} (?) ($\forall n$?)

Examples.

(1) $(G_n = \mathbb{Z}/p^n\mathbb{Z}_S)_n = \mathbb{Q}_p/\mathbb{Z}_p$ of height 1.

(2) $(G_n = \mu_{p^n\,S})_n = \mu_{p^\infty}$ of height 1

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(3) $A \to S$ abelian scheme of dim g.

$$(G_n = A[p^n])_{n\geq 0} = A[p^\infty] \quad \text{of height } 2g$$

Over a field k of char 0 $(S = \operatorname{Spec} K)$:

G p-div. gp./K of height h.

$$G \times_k \bar{K} \cong (\mathbb{Q}_p/\mathbb{Z}_p)^h_{\bar{K}}.$$

$G_n \cong (\mathbb{Z}/p^n\mathbb{Z})^h$, $\quad G/K \rightsquigarrow G_K$ representation on G_n.

$G_K \to$ ~~$GL_h(\mathbb{Z}/p^n\mathbb{Z})$~~ $GL_h(\mathbb{Z}/p^n\mathbb{Z})$ for all n.

Over a field k of char p:

$|G_n(\bar{k})|$ need not equal p^{ng}.

$\bar{k} \to \bar{k}[t]/(t^{p^n}-1)$. $\quad |G_n(\bar{k})| = 1 \iff G$ connected.

$G = \mathbb{Q}_p/\mathbb{Z}_p$, $\quad |G_n(\bar{k})| = p^n \Rightarrow G$ étale.

Lem. 1. Over a finite field k, any p-div. group is isomorphic to $G^{inf} \oplus G^{et}$, with G^{inf} connected & G^{et} etale.

E/k ell. curve ; $E[p^\infty]$.

$E \to$ supersingular $\quad |E[p](\bar{k})| = 1$

$E \to$ ordinary $\quad |E[p](\bar{k})| = p$

E supersingular $\to E[p^\infty]$ is connected,

E ordinary $\Rightarrow E[p^\infty]$ isomorphic $(/\bar{k})$

Generalisation of Lem. 1. R : henselian local ring$/k$, G p-div. gp$/R$. Then there is an exact sequence

$$0 \to G^{inf} \to G \to G^{ét} \to 0$$

where G^{inf} is connected$(/R)$ & $G^{ét}$ is étale.

Rk. Doesn't split in general. (not clear what the morphisms are, though.)

$$\left(0 \to E[p^\infty]^{inf} \to E[p^\infty] \to E[p^\infty]^{ét} \to 0\right)$$

E ordinary ell. curve$/\mathbb{Z}_p$.

Formal groups. R complete DVR of res. char. p.

Recall : a 1-dim$^{\underline{l}}$ formal group$/R$ is $F \in R[[X,Y]]$ satisfying some axioms.

$$[p^n]_F \in R[[X]]$$

$$\left(R[[X]]/[p^n]_F \right)_n$$ has the structure of a p-div. gp.$/R$.

Any formal p-div. gp is connected over R.

Conversely, any connected p-divisible group$/R$ is formal.

$K/\mathbb{Q}_p$ finite extension. $\mathcal{O} = \mathcal{O}_K$.
($k/\mathbb{F}_p$ corresponding)
$R = \mathcal{O}_K$-alg, π uniformiser.

A π-divisible $\mathcal{O}$-module$/R$ is a p-divisible group G over R together with $\iota : \mathcal{O}_K \to \mathrm{End}_R(G)$ (hom of $\mathbb{Z}_p$-algs?) (thought of as limit?) s.t. $\mathcal{O}_K \to \mathrm{Lie}(G_n)$ agrees with the $\mathcal{O}_K$-action coming from $\mathcal{O}_K$-alg. structure on R.

(Can define p-div. gps $/W(k)$) ← Witt

Notation. $k, K, \mathcal{O}$ as above. For any k'/k, define

$$W_K(k') := W(k') \underset{W(k)}{\otimes} \mathcal{O}_K.$$
(max. unram. extension of $\mathcal{O}_K$ with res. field k')

Level n structures.

Idea: $A \to K$ field of char 0 (A abv. variety)

A level N-structure is a choice of a basis of $A[N]$.

Defn for formal groups.

If $F \in R[[x,y]]$ formal ~~group~~ $\mathcal{O}$-mod. of height h, giving a level π^N-structure is the same as giving

$$\varphi : (\mathcal{O}/\pi^N)^h \to R[[x]]/[\pi^N]_F ,$$

$$\prod_{x \in (\mathcal{O}/\pi^N)^h} (T - \varphi(x)) \,\Big|\, [\pi^N]_F(T)$$

(Or, Scholze:) A level π^N-structure on a π-divisible $\mathcal{O}$-mod G over R of height h is:

$$X_1, \ldots, X_h : (\mathcal{O}/\pi^N)_R \to G[\pi^N]$$

$$\text{s.t.} \sum_{(i_1,\ldots,i_h) \in (\mathcal{O}/\pi^N)^h} [i_1 X_1 + \cdots + i_h X_h] = [G[\pi^N]]$$

as relative Cartier divisors on G/R.

<u>Idea</u>: classify π-div. $\mathcal{O}$-mods over finite fields.
Study deformations of such $\mathcal{O}$-mods over $W(k)$-algs.

$\rightsquigarrow$ moduli space M_0

- level structures $\rightsquigarrow M_n \to M_0$ Galois with group $GL_n(\mathcal{O}/\pi^h\mathcal{O})$.

Classification over finite fields.

<u>Thm.</u> The category of π-div. $\mathcal{O}$-modules over $\bar{k}$ is equivalent to the category of (M, F, V):

- M a free $W_K(k')$-module of finite rk
- F σ-linear: $(\sigma \in Gal(k'/k) \rightsquigarrow \sigma \in Aut_{W(k)}(W(k')))$

$$F_c = \sigma(c)F \qquad (c \in W(k'))$$

- V σ^{-1}-linear: $cV = V\sigma(c)$
- $FV = VF = \pi_K \cdot 1_M$. ($M$: Dieudonné module of G)

Structure theory.

<u>Proposition.</u> $\forall\ 0 < k \leq h$, $\exists!$ π-divisible $\mathcal{O}_K$-mod$/\bar{k}$ M (up to isom.) of $\mathcal{O}_K$-height h & étale $\mathcal{O}_K$-height k.
It is the Dieudonné module with basis

$$(d_1, \dots, d_k, e_1, \dots, e_{h-k})$$

$$Vd_i = d_i$$
$$Ve_{h-k} = \pi_K \cdot e_1$$

Classification / k

$$(M,F,V)_{/W(k)} \longrightarrow (M,F,V)_{/W(\bar{k})}$$

π-div. $\mathcal{O}$-mod of height h over $W(k)$
$\leftrightarrow GL_n(\mathcal{O})$ conjugacy classes

$\beta \in GL_n(\mathcal{O}) \,\, \begin{pmatrix} \pi & & \\ & \ddots & \\ & & 1 \end{pmatrix}^{GL_n(\mathcal{O})}$

$GL_k(\mathcal{O}) \begin{pmatrix} \pi & & \\ & \ddots & \\ & & 1 \end{pmatrix} GL_k(\mathcal{O}) \times GL_{h-k}(\mathcal{O})$

Deformations of π-div. $\mathcal{O}$-mods.

$\mathcal{G}$ / finite field k. $\mathcal{C} = $ (Artinian local $W(k)$-algs).

$$M_0 : R \longmapsto \begin{pmatrix} G \to R \quad p\text{-div. gp.} \\ \text{with isom } G \otimes k \cong \mathcal{G} \end{pmatrix} \Big/ \text{isom.}$$

Thm. $\mathcal{G}$ is representable (formally), & over $\bar{k}$ it's isomorphic to $\mathrm{Spf}\ W(\bar{k})[[u_1, \dots, u_{n-1}]]$.

$$\mathcal{G} \cong \mathcal{G}^{inf} \oplus (K/\mathcal{O})^k$$

Deformations of $\mathcal{G}^{inf}$ over $R \longleftrightarrow$ formal gps $F \in R[[X,Y]]$
s.t. $\bar{F}$ is $\underset{(\cong)}{}$ the formal gp of $\mathcal{G}^{inf}$

Deformations of $\mathcal{G}^{inf}$ over $\bar{k}$ are represented by

$$E_0 \cong W(\bar{k})[[t_1, \dots, t_{n-k}]].$$

If F is a deformation of $\mathcal{G}^{inf}$ over R, then (one shows) extensions to deformations of $\mathcal{G}$ are in bijection with k-tuples $(\varphi_1, \dots, \varphi_k) : R \to F$. $\rightsquigarrow M_0 \cong E_0[[t_1, \dots, t_k]]$

More generally, we can do this with level structures:

$(\mathcal{G}, \varphi)$ over k

(rep'd by:) $M_n \to M_{n-1} \to \cdots \to M_0$.

<u>Properties of these maps:</u> $M_n \to M_0$ is Galois with group $GL_n(\mathcal{O}/\pi^n\mathcal{O})$.

$\rightsquigarrow$ associated Raynaud generic fibres

$$M_n \times K \to M_0 \times K \text{ étale.}$$

A / finite field k (abv. variety)

$N_0 : \mathcal{C} \to (\tilde{A} \to R$ abelian with fibre $A)$ / isom.

$A \mapsto A[p^\infty]$

$(\tilde{A} \to R)$ $(\mapsto ?)$ $(\tilde{A}[p^\infty] \to R)$ $\to N_0$ is representable by a formal scheme (N_0), &

<u>Thm.</u> (Serre–Tate.) $(\tilde{A} \to R) \longrightarrow (\tilde{A}[p^\infty] \to R)$ gives an isom $M_0 \xrightarrow{\sim} N_0$.