① <u>Shimura varieties and Rapoport-Zink spaces</u> P. Chojecki 28/5/15

Recall there is a notion of <u>Shimura datum</u> (G,X)
$\rightsquigarrow Sh_K(G,X)$ a <u>Shimura variety</u> s.t.

$$Sh_K(G,X)(\mathbb{C}) \cong G(\mathbb{Q})\backslash G(\mathbb{A}_f)\times X / K$$

Here $K \subseteq G(\mathbb{A}_f)$ is a compact open subgroup.

<u>Notations</u> $\mathbb{F}_0$ – totally real field, $[\mathbb{F}_0:\mathbb{Q}]$ even,
τ infinite place of $\mathbb{F}_0$, x_0 finite place of $\mathbb{F}_0$
$\mathbb{K}$ – imaginary quadratic field, $\mathbb{F} = \mathbb{F}_0\cdot\mathbb{K}$.
(rational prime below x_0 split in $\mathbb{K}$).

$\mathbb{K} \hookrightarrow \mathbb{C}$, n integer
$x_0 = x\cdot x^c$ in $\mathbb{F}$,
$\mathbb{D}$ central division algebra over $\mathbb{F}$ dimension n^2

Assume: – have an involution on $\mathbb{D}$ (of the second kind).
– $\exists\, h_0 : \mathbb{C} \to \mathbb{D}_{\mathbb{R}}$

<u>Lemma 8·1</u> $\exists$ group $G_0(R) = \{g \in (\mathbb{D}\otimes_{\mathbb{F}_0} R)^\times \mid gg^* = 1\}$
such that G_0 is a unitary group of signature $(1, n-1)$ at τ, and $(0,n)$ at all the other infinite places.

$$G = G_0/\mathbb{Q}, \quad G_{\mathbb{C}} = \Big(\prod_{\mathbb{F}_0 \hookrightarrow \mathbb{C}} GL_n\Big)\times \mathbb{G}_m.$$

Put $Sh_K = Sh_K(G,X)$ (X = conj. class of h_0, $K \subset G(\mathbb{A}_f)$ compact open subgroup)

②

Let $\xi = (V, \pi)$ be an irred. alg. rep of G.

By considering

$$\begin{array}{c} G(\mathbb{Q}) \backslash G(\mathbb{A}_f) \times \cancel{X} \times V / K \\ \downarrow \\ G(\mathbb{Q}) \backslash G(\mathbb{A}_f) \times X / K \end{array},$$

we get an ℓ-adic local system $\mathcal{L}_{\xi, K}$ on Sh_K.

Consider $H^*_\xi = \varinjlim_K H^*(Sh_K, \mathcal{L}_{\xi, K})$.

Get $[H_\xi]$ — alternating sum of H^*_ξ in Grothendieck group of $\mathrm{Gal}(\overline{F}/F) \times G(\mathbb{A}_f)$ representations

continuous — admissible smooth

<u>Goal</u> compute traces of certain functions on $[H_\xi]$.

Take $F/\mathbb{Q}_p$ finite, $\mathcal{O}_F = \mathcal{O} \subset F$.

We want to construct a model of Sh_K over $\mathcal{O}_F$.

We'll define a <u>functor</u> M_K on $\mathcal{O}$-schemes,

- prove it's representable
- $M_K(\mathbb{C}) \cong Sh_K(\mathbb{C})$.

(3)

adeles away from p

Data: $K^p \subset G(\mathbb{A}_f^p)$,

$$K_p^m = \left(1 + \varpi^m M_n(\mathcal{O})\right) \times \left(1 + p^m \mathcal{O}_D\right) \times \mathbb{Z}_p^\times$$

$\cap$

$G(\mathbb{Q}_p)$ $\qquad B := \underline{D} \times D^{op}$.

Fixed: $K^p \subset G(\mathbb{A}_f^p)$ compact open subgroup

$K_p^m \subset G(\mathbb{Q}_p)$ m-congruence subgroup.

Define

$$M_{K^p . K_p^m} : \left\{\begin{matrix} S\text{-locally} \\ \text{Noetherian} \\ \text{over } \mathcal{O} \end{matrix}\right\} \longmapsto \left\{(A, \lambda, \iota, \eta^p, \eta_p)\right\}\Big/\sim$$

where:
- A is a projective abelian scheme over S (up to prime-to-p isogeny) of dimension n
- polarisation $\lambda : A \longrightarrow A^\vee$
- $*$-homomorphism $\iota : \mathcal{O}_D \longrightarrow \mathrm{End}(A)$ satisfying the "Kottwitz determinant condition"
- level structure η^p (due to Kottwitz)
- η_p is m-level structure on $A[p^\infty]$ (as in week 3)

Equivalence relation

$(A, \lambda, \iota, \eta^p, \eta_p) \approx (A', \lambda', \iota', \eta^{p\prime}, \eta_p') \iff$

$\exists$ isogeny $A \longrightarrow A'$ which carries λ to a $\mathbb{Z}_{(p)}^\times$-multiple of λ', η^p to $\eta^{p\prime}$, η_p to η_p'.

Proposition (Kottwitz, Deligne, ...)
For K^P sufficiently small, M_{K^P, K_p^m} is representable by a projective scheme
$$Sh_{K^P,m}/\mathcal{O}$$
$(- \times_{\mathcal{O}} F)$

Remark Generic fibre of $Sh_{K^P,m}$ is a finite disjoint union of $Sh_{K^P K_p^m}/\mathbb{F}_\omega$ where ω is a place s.t. $\mathbb{F}_\omega = F$.

Lemma 9.2 Let x be a closed point on $Sh_{K^P,0}(\bar{k})$, where $k = \mathcal{O}/\omega$ is the residue field of $\mathcal{O}$, $\bar{k}$ is an algebraic closure of k.

$x \rightsquigarrow A_x \rightsquigarrow H_x = A_x[p^\infty]$, a p-divisible group over $\bar{k}$.

Then
$$\widehat{\mathcal{O}_{Sh_{K^P,0},x}} \cong R_{H_x,0}$$
where $R_{H_x,0}$ is the deformation ring parametrizing deformations of H_x.

Proof Serre-Tate theorem:
"deforming A" = "deforming $A[p^\infty]$" $\square$

Extend this resultto to level m:

Let $\pi_m : Sh_{K^p,m} \longrightarrow Sh_{K^p,0}$.

Then $\pi_m^{-1}\left(Spf\ \widehat{\mathcal{O}_{Sh_{K^p,0},x}}\right) \cong Spf\left(R_{H_x,m}\right)$

Here $R_{H_x,m}$ parametrises deformations of H_x with m-level structures.

How will we use it (in Week 7)?

For any $\tau \in Frob^r \cdot I_F \subseteq W_F$, $(**)$ and any $h \in C_c^\infty(GL_n(\mathcal{O}))$, we'll define $f_{\tau,h} \in C_c^\infty(GL_n(F))$ using vanishing cycles (week 6) on

$$\varprojlim_m Spf\left(R_{H_x,m}\right).$$

Recall: Scholze's characterisation of LLC:

$\boxed{\text{LLC for } GL_n(F)}$ For any irred smooth rep π of $GL_n(F)$, $\exists!$ n-dimensional rep $rec(\pi)$ of W_F, such that for all τ, h as above, we have

$$tr\left(f_{\tau,h}\right) = tr\left(\tau \mid rec(\pi)\right)\, tr\left(h,\pi\right).$$

1st step: embed it into a global situation

⑥

Thm 10.2 (Harris–Taylor) ✓ Assume that π is an irreducible smooth rep of $GL_n(F)$ that is either essentially square-integrable, or a generalised Speh representation.

Then there is an irreducible admissible rep π_f of $G(\mathbb{A}_f)$, and an irreducible alg. rep ξ of $G(\cancel{\mathbb{A}_f})$ such that

(1) $W^*_\xi(\pi) = \cancel{\varinjlim_K H^*(Sh_K}$,

$$\mathrm{Hom}_{G(\mathbb{A}_f)}(\pi_f, H^*_\xi) \neq 0$$

for some $*$.

Here $H^*_\xi = \varinjlim_K H^*(Sh_K, \mathcal{L}_{\xi,K})$

(2) the component π_w of π_f is an unramified twist of π.

For the following one uses Kottwitz's result, + Lemma 9.2.

Corollary 10.1 Assume π_f is an irred. adm. rep. of $G(\mathbb{A}_f)$ with $W^*(\pi_f) \neq 0$.

Recall $G(\mathbb{Q}_p) = GL_n(F) \times D'^\times \times \mathbb{Q}_p^\times$.

Write $\pi_{f,p} = \pi_w \otimes \pi_p^w \otimes \pi_{p,0}$.

Assume that $\pi_{p,0}$ is unramified, and let $\chi_{\pi_{p,0}}$ be the character of W_F corresponding to $\pi_{p,0}$ by local Class Field Theory.

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Then, for any τ, h as in $(**)$, we have $\left(f^\vee(g) := f((g^{-1})^t)\right)$

$$\mathrm{tr}\left(f^\vee_{\xi,h} \,\middle|\, \pi_w\right) = \frac{1}{a(\pi_f)} \mathrm{tr}\left(\tau \,\middle|\, [W_\xi(\pi_f)] \otimes \chi_{\pi_{p,0}}\right) \cdot \mathrm{tr}\left(h^\vee \,\middle|\, \pi_w\right).$$

Here $\dim [W_\xi(\pi_f)] = a(\pi_f) \cdot n$.

$\rightsquigarrow$ one can define in the Grothendieck group of W_F-reps

$$\mathrm{rec}(\pi) := \frac{1}{a(\pi_f)} [W_\xi(\pi_f)]^\vee \otimes \chi^{-1}_{\pi_{p,0}} \otimes \chi^{-1}$$

here $\pi_w = \pi \otimes \chi$ for some unramified character χ (LCFT).

<u>Still to prove</u> (1) $\mathrm{rec}(\pi)$ is effective (week 7)

(2) $\mathrm{rec}(\pi)$ gives a bijection (week 8).