Nearby cycles ~~and~~

1. Nearby cycles for complex analytic spaces

$f: X \longrightarrow D$ proper morphism of cx analytic spaces.

$D = \{z \in \mathbb{C} : |z| \leq 1\}$, $D^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$.

$X_t := f^{-1}(t)$.

$X_0 :=$ special fibre

Variant of Thom's Isotopy Theorem (see SGA 7. XIV. 1.3.5)

After shrinking D, $f|_{f^{-1}(D^*)}: f^{-1}(D^*) \longrightarrow D^*$ is a topological fibration.

Suppose further $r: X \longrightarrow X_0$ is a retraction, s.t. $\forall$ open connected $U \subseteq D^*$, $\exists$ isom

$$\Gamma_U : f^{-1}(U) \xrightarrow{\sim} X_a \times U \qquad (a \in U)$$

s.t. $r\Gamma_U^{-1}(x, u)$ only depends on x. [$(\Gamma_U)_U$ is 'compatible']

Then for any $t \in D^*$ and $\varphi : [0,1] \longrightarrow D^*$, $x \longmapsto e^{2\pi i x} t$,

we get a monodromy operator

$$T : X_t = (\varphi^* X)_0 \xrightarrow[(\Gamma)]{\sim} (\varphi^* X)_1 = X_t .$$

Fact

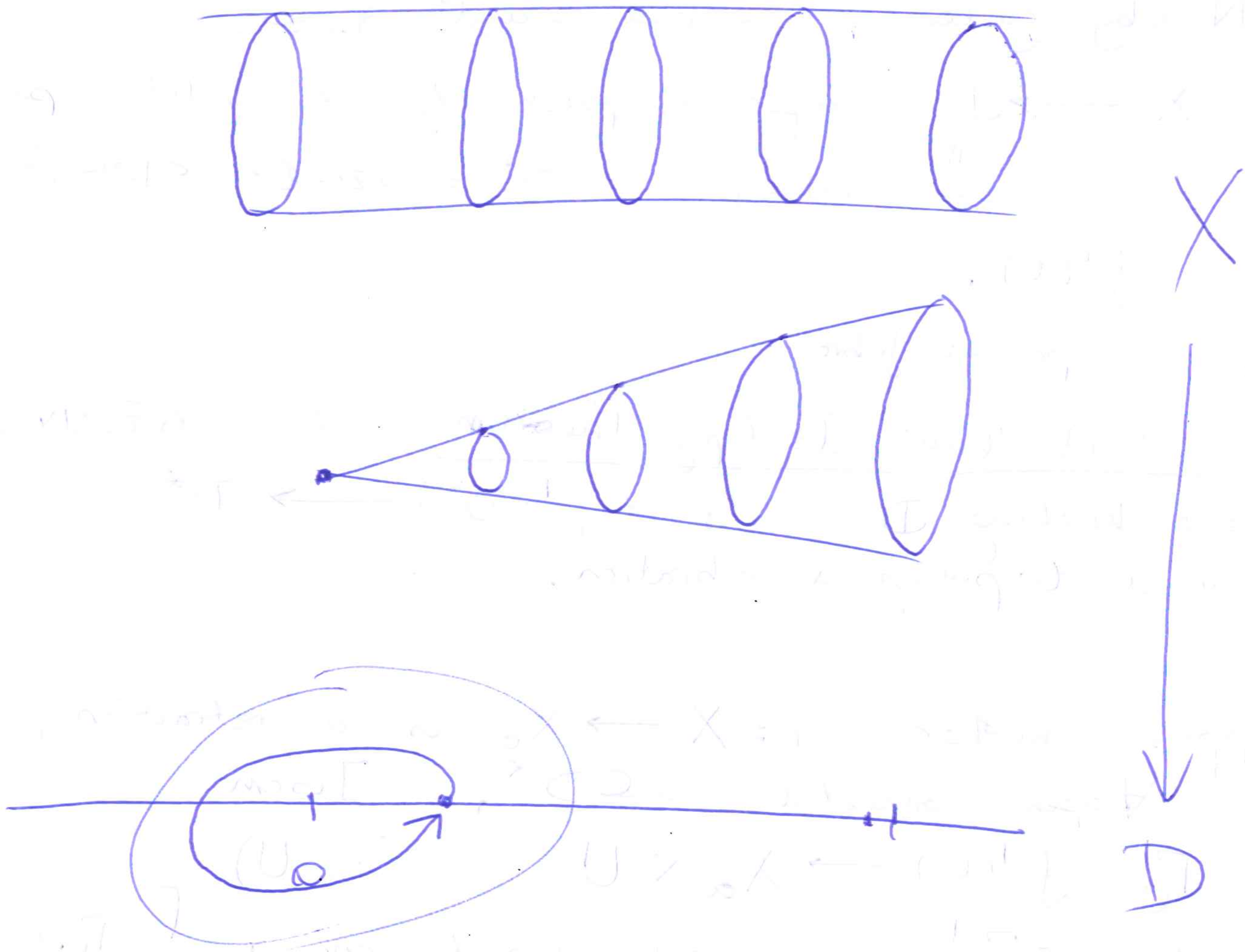
- One can reconstruct (X, f) from $(X_t, X_0, T, r|_{X_t})$.

Let $\Psi := (r, f) : X \longrightarrow X_0 \times D$.

Higher direct images $(i > 0)$

$(R^i \Psi_* \mathbb{Z})_t$ deserve the name

vanishing cycles.


X
D

Deligne's nearby cycles X – good connected topological space,

$f : X \to D$ continuous.

$\widetilde{D^*} \xrightarrow{p'} D^*$ universal covering of D^*

$\tilde{D} = \widetilde{D^*} \cup \{0\}$ with topology s.t.

- $$\begin{array}{ccc} \tilde{D} & \overset{\bar{j}}{\hookleftarrow} & \widetilde{D^*} \\ p \downarrow & & \downarrow p' \\ D & \overset{j}{\hookleftarrow} & D \end{array}$$ commutes,
- $\bar{j}$ is an open embedding,
- $p(0) = 0$
- $\{p^{-1}(U) : U \ni 0 \text{ open}\}$ forms a base of open nhoods of 0 in $\tilde{D}$

$$\bar{X} := X \times_D \tilde{D},$$

$$\bar{X}^* := X \times_D \widetilde{D^*} = \bar{X} \setminus X_0.$$

This gives the commutative diagram

$$\begin{array}{ccccc} X_0 & \xrightarrow{\bar{i}} & \bar{X} & \xleftarrow{\bar{j}} & \bar{X}^* \\ \| & & \downarrow p & & \downarrow p' \\ X_0 & \xrightarrow{i} & X & \xleftarrow{j} & X^* \end{array}$$

Defn Let $\mathcal{F}$ be a sheaf on X.
The **nearby cycles sheaf** is the triple

$$(i^*\mathcal{F}, \bar{i}^* \bar{j}_*((jp')^*\mathcal{F}), \alpha) =: \Psi(\mathcal{F})$$

$$\in X_0 \times \mathcal{D} \quad \text{(fibre-product of toposes)}$$

Here $Ob(X_0 \times \mathcal{D}) := \{ (F_0, F_t, \alpha) :$
$F_0 \in Sh(X_0)$, $F_1 \in Sh(X_0)$ with an action of $\pi_1(D^*)$; $\alpha : F_0 \longrightarrow F_1^{\pi_1(D^*)} \}$

Get $\boxed{R\Psi : D^+(X) \longrightarrow D^+(X_0 \times \mathcal{D})}$.

and $\boxed{R\Psi_\eta : D^+(X^*) \longrightarrow D^+(X_0 \times \mathcal{D})}$

<u>Application</u> : nearby cycles are used to relate the cohomology of the generic fibre with that of the special fibre.

e.g. for $\mathcal{K} \in D^+(f^{-1}(D^*))$,

$$\boxed{\varinjlim_{U \ni 0} \mathbb{H}^i(X_{U^*}, \mathcal{K}) \xrightarrow{\sim} \mathbb{H}^i(X_0, R\Psi_{\bar\eta}(\mathcal{K}))}$$

$$(X_{U^*} = X \times_D p^{-1}(U))$$

$$(\Psi_{\bar\eta}(\mathcal{F}) = \bar{i}^* \bar{j}_* (p'^* \mathcal{F})).$$

2. Nearby cycles for schemes

Basic analogy: $\mathbb{D} \longleftrightarrow \operatorname{Spec}(R) =: S$

where R is a henselian local ring. $s \in S$ – closed point.

Let $p : X \longrightarrow S$ be an S-scheme.

Let $\overline{X} := X \underset{S}{\overline{\times}} \overline{S}$ (a topos).

$\operatorname{Ob}(X \underset{S}{\overline{\times}} \overline{S}) = \{ (\mathcal{F}_{\bar{s}}, \mathcal{F}_{\bar{\eta}}, \varphi) :$

$\mathcal{F}_{\bar{s}} \in \operatorname{Sh}(X_{\bar{s}})$

$\mathcal{F}_{\bar{\eta}} \in \operatorname{Sh}(X_{\bar{\eta}})$ + $\operatorname{Gal}(\bar{\eta}/\eta)$-action; $I(\bar{\eta}/\eta)$

$\varphi : \mathcal{F}_{\bar{s}} \longrightarrow \mathcal{F}_{\bar{\eta}}$ $\}$.

Get diagram

$$\begin{array}{ccccc} X_{\bar{s}} & \xhookrightarrow{\bar{i}} & \overline{X} & \xhookleftarrow{\bar{j}} & X_{\bar{\eta}} \\ \downarrow & & \downarrow & & \downarrow p \\ X_s & \xhookrightarrow[i]{} & X & \xhookleftarrow[j]{} & X_\eta \end{array}$$

Cartesian over

$$\begin{array}{ccccc} \bar{s} & \hookrightarrow & \overline{S} & \hookleftarrow & \bar{\eta} \\ \downarrow & & \downarrow & & \downarrow \\ s & \hookrightarrow & S & \hookleftarrow & \eta \end{array}$$

Defn The <u>nearby cycles functor</u>

$$\Psi : (X_{\acute{e}t})^{\sim} \longrightarrow (X_s \underset{s}{\times} S)^{\sim}$$

sends $\mathcal{F}$ to $(\mathcal{F}_s, \bar{i}^* \bar{j}_* p^* \mathcal{F}_\eta, \varphi)$.

Here $X_s \times_s S$ is the topos with objects

(F_s, F_η, φ)

where:
- $F_s \in Sh(X_{s,ét})$ viewed as a $Gal(\bar{s}/s)$-equivariant sheaf on $X_{\bar{s}} = X_s \times_s \bar{s}$
- F_η ~~$\in Sh(X_{\bar{s}} \ldots)$~~ is a $Gal(\bar{\eta}/\eta)$-equivariant étale sheaf on $X_{\bar{s}}$,
- $\varphi : F_{\bar{s}} \longrightarrow F_{\bar{\eta}}$ is an equivariant map.

Get $R\Psi : D^+(X_{ét}) \longrightarrow D^+(X_s \times_s S)$.

This comes with a distinguished triangle

$$\mathcal{F}|X_{\bar{s}} \longrightarrow R\Psi(\mathcal{F}) \longrightarrow R\Phi(\mathcal{F}) \xrightarrow{+1}$$

Defn $R\Phi(\mathcal{F})$ is the complex of vanishing cycles of $\mathcal{F}$.

3. Nearby cycles for formal schemes and Berkovich spaces

k – non-Archimedean field

k° – ring of integers of k Move to p. 7

k°-$\mathfrak{F}sch$:= cat. of formal schemes locally finitely presented over $Spf(k^\circ)$.

$\tilde{k} = k^\circ / k^{\circ\circ}$ – residue field of k.

Lemma Let $\mathfrak{X} \in k^\circ$-$\mathfrak{F}sch$.

$$\left\{ \begin{array}{c} \text{formal schemes} \\ \text{étale over } \mathfrak{X} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{schemes} \\ \text{étale over } \mathfrak{X}_s \end{array} \right\}$$

$\mathcal{Y} \longmapsto \mathcal{Y}_s$ is an equivalence.

⑥ Let X be a (Berkovich) k-analytic space. It's a compact topological space, so have topos $X^\sim$.

Berkovich also defined a ~~richer~~ richer G-topology X_G on X; and a morphism of ~~sites~~ toposes

$$(\pi_*, \pi^*) : X_G^\sim \longrightarrow X^\sim .$$

<u>Fact</u> 1) π^* is fully faithful, but π_* is <u>not</u>.

2) If X is a rigid analytic space, $r(X)$ is the associated Huber space, $a(X)$ ——— Berkovich space, then $\exists\, r(X) \twoheadrightarrow a(X)$, and $a(X)$ is the maximal Hausdorff quotient of $r(X)$.

We have $X^\sim \cong r(X)^\sim$

and $X^\sim \cong a(X)_G^\sim$.

Similarly, Berkovich defined the <u>quasi-étale</u> site $X_{q\acute{e}t}$ (for any k-analytic space X), together with

$$(\mu_*, \mu^*) : X_{q\acute{e}t}^\sim \longrightarrow X_{\acute{e}t}^\sim .$$

<u>Fact</u> $\forall \mathcal{F} \in X_{\acute{e}t}^\sim$, $\mathcal{F} \xrightarrow{\sim} \mu_* \mu^* \mathcal{F}$.

So μ^* is fully faithful.

(7) Now let $\mathfrak{X} \in k^\circ\text{-}\mathcal{F}sch$.

By the Lemma, we can fix a functor

$$\mathcal{Y}_s \longmapsto \mathcal{Y}$$

inverse to "special fibre" from the Lemma.

This gives a morphism of sites

$$\nu : (\mathfrak{X}_\eta)_{ét} \longrightarrow (\mathfrak{X}_s)_{ét}$$

$$\mathcal{Y} \longleftarrow \mathcal{Y}_s .$$

Get $(\mathfrak{X}_\eta)^\sim_{ét} \xrightarrow{\mu^*} (\mathfrak{X}_\eta)^\sim_{qét} \xrightarrow{\nu_*} (\mathfrak{X}_s)^\sim_{ét}$

<u>Defn</u> $\Theta := \nu_* \mu^*$ "a specialisation functor".

Berkovich calls it "nearby cycles".

<u>Thm</u> For $\mathcal{F}$ an abelian étale sheaf on $\mathfrak{X}_\eta$, there is a spectral sequence

$$E_2^{p,q} = H^p(\mathfrak{X}_s, (R^q\Theta)(\mathcal{F})) \Rightarrow H^{p+q}(\mathfrak{X}_\eta, \mathcal{F}).$$

<u>Defn</u> Let $\mathfrak{X} \in k^\circ\text{-}\mathcal{F}sch$, $\overline{\mathfrak{X}} := \mathfrak{X} \hat{\otimes}_{k^\circ} (\widehat{k^s})^\circ$;

then $\mathfrak{X}_{\bar{s}} = \mathfrak{X}_s \otimes k^s$ and

$\mathfrak{X}_{\bar{\eta}} = \mathfrak{X}_\eta \hat{\otimes} \widehat{k^s}$; the ~~vanishing~~ <u>nearby cycles functor</u>

$$\Psi_\eta : \mathfrak{X}^\sim_{\eta_{ét}} \longrightarrow \mathfrak{X}^\sim_{\bar{s}ét}$$

sends $\mathcal{F} \longmapsto \nu_{K*}\mu_K^* \mathcal{F}_K$

where $K = \widehat{k^s}$.

<u>Fact</u> • there is a ~~compatible~~ canonical $G_\eta = \mathrm{Gal}(k^s/k)$ - action on $\Psi_\eta(\mathcal{F})$, compatible with the $G_s = \mathrm{Gal}(\bar{k}^s/\bar{k})$ — action on $\mathcal{X}_{\bar{s}}$.

• For an abelian étale sheaf $\mathcal{F}$ on $\mathcal{X}_\eta$,

$$\Psi_\eta(\mathcal{F}) = \varinjlim_K \bar{i}_K^*(\Theta_K(\mathcal{F}_K))$$

$$\left(k \underset{\text{finite}}{\subset} K \subset k^s \ , \ \underset{\text{canonical}}{\bar{i}_K : \mathcal{X}_{\bar{s}} \to \mathcal{X}_{s_K}} \right)$$

• The action of G_η on $\Psi_\eta(\mathcal{F})$ is continuous.

<u>Thm</u> $\exists$ spectral sequence

$$E_2^{p,q} = H^p(\mathcal{X}_{\bar{s}}, R^q\Psi_\eta(\mathcal{F})) \Longrightarrow H^{p+q}(\mathcal{X}_\eta, \mathcal{F}).$$

<u>Proof</u>

$$\begin{array}{ccc} D^+(\mathcal{X}_\eta) & \xrightarrow{R\Psi_\eta} & D^+_{G_\eta}(\mathcal{X}_{\bar{s}}) \\ & {\scriptstyle R\Gamma}\searrow \quad \swarrow{\scriptstyle R\Gamma} & \\ & D^+(\mathrm{Ab}) & \end{array}$$

commutes. □

($D^+_{G_\eta}(\mathcal{X}_{\bar{s}})$:) derived category of étale abelian G_η – sheaves on $\mathcal{X}_{\bar{s}}$.

Prop (Berkovich) ($\mathfrak{X}$ – special formal scheme)

Let $\mathcal{Y}$ be a closed subset of $\mathfrak{X}_s$, and let $\mathcal{F}^\bullet \in D^+(\mathfrak{X}_\eta)$. Then

$$R\Gamma_{\mathcal{Y}_{\bar{K}}}\left(\mathfrak{X}_{s_{\bar{K}}}, R\Theta_{\bar{K}}(\mathcal{F}^\bullet)\right) \cong R\Gamma_{\pi^{-1}(\mathcal{Y})_{\bar{K}}}\left(\mathfrak{X}_{\eta_{\bar{K}}}, \mathcal{F}^\bullet\right)$$

Here $\pi : \mathfrak{X}_\eta \longrightarrow \mathfrak{X}_s$ is the reduction map.

Cor If $\mathfrak{X}$ is lft over k°, and $\mathcal{Y}$ is quasicompact, then

$$R\Gamma_{\mathcal{Y}_{\bar{K}}}\left(\mathfrak{X}_{s_{\bar{K}}}, R\Theta_{\bar{K}}(\mathcal{F}^\bullet)\right) \cong R\Gamma_c\left(\pi^{-1}(\mathcal{Y})_{\bar{K}}, \mathcal{F}^\bullet\right).$$

Comparison Theorem (Berkovich)

$\mathcal{S} = \mathrm{Spec}(k^\circ)$, k° – local Henselian DVR

$\mathcal{X} \longrightarrow \mathcal{S}$ scheme, locally of finite type

$\widehat{\mathcal{X}_{/\mathcal{Y}}}$ — formal completion of $\mathcal{X}$ along ~~closed~~ subscheme $\mathcal{Y} \subseteq \mathcal{X}_s$.

Then $\left(\widehat{\mathcal{X}_{/\mathcal{Y}}}\right)_\eta$, the Raynaud-Berthelot generic fibre

is isomorphic to $\pi^{-1}(\mathcal{Y})$

Thm Let $\mathcal{F}$ be an étale abelian constructible sheaf on $\mathfrak{X}_\eta$ with torsion orders prime to $\mathrm{char}(\tilde{k})$.

Then $R^q\Theta(\mathcal{F})\big|_{\mathcal{Y}} \simeq R^q\Theta(\widehat{\mathcal{F}_{\mathcal{Y}}})$, and

PTO

$$(R^q \Psi_\eta)(\mathcal{F})\big|_{\bar{y}} \simeq (R^q \Psi_\eta)(\widehat{\mathcal{F}/y}) .$$

$$\begin{array}{ccc} \mathcal{X} & \rightsquigarrow & \mathcal{X}_\eta \\ \downarrow & & \downarrow \\ \widehat{\mathcal{X}} & \rightsquigarrow & \widehat{\mathcal{X}}_{\eta/y} \end{array}$$