Goal: (1) define functions

$f_{\tau, h}$:

(2) Define $\operatorname{rec}(\pi)$ as a virtual repn, & prove it's effective.

§1 Deformations

$F|\mathbb{Q}_p$ fin, $\mathcal{O} = \mathcal{O}_F$, ϖ, κ.

Let $F_r|F$ unram extn of degree r. $\mathcal{O}_r$. κ_r.

For any $\beta \in GL_n(\mathcal{O}_r) \begin{pmatrix} \varpi & & \\ & \ddots & \\ & & 1 \end{pmatrix} GL_n(\mathcal{O}_r)$

$\rightsquigarrow \overline{H}_\beta$

$\overline{H}_\beta$ is a ϖ-div 1-dim $\mathcal{O}$-module of height h over κ_r.

$\overline{H}_\beta$ is defined by specifying the Dieudonné module of $\overline{H}_\beta$.

(M, F, V): $M = \mathcal{O}_r^n$, $F = \beta \circ \sigma \curvearrowright \mathcal{O}_r^n$.

here σ is the usual Frob of F_r/F.

Let R_β be the ring of formal deformations of $\overline{H}_\beta$.

Let $R_{\beta, m}$ be the ring parametrizing deformations of $\overline{H}_\beta$ with Drinfeld m-lev. structure.

Prop 2.3: (1) R_β is formally smooth, complete Noeth $\mathcal{O}$-~~module~~ algebra

(2) $R_{\beta, m}$ is regular & $R_{\beta,m}|R_\beta$ is Galois w/ gp $GL_n(\mathcal{O}/\varpi^m\mathcal{O})$.

§2 Vanishing cycles:

R DVR, s special pt, η generic point.

formal scheme

$$\begin{tikzcd} X_s \arrow[r, hook, "i"] \arrow[d] & X \arrow[d] & X_\eta \arrow[l, hook', "j"'] \arrow[d] \\ s \arrow[r] & \mathrm{Spf}\, R & \eta \arrow[l, hook'] \end{tikzcd}$$

Vanishing cycles for a sheaf $\mathcal{F} \in \tilde{X}_{\eta, et}$

"$R\Psi_X \mathcal{F} = R i^* j_* \mathcal{F}$"

(but topological issues)

[Berkovich]

§3 Functions

Define $R\Psi_\beta = \varinjlim_m H^0(R\Psi_{\mathrm{Spf}\, R_{\beta,m}} \overline{\mathbb{Q}}_\ell)$

<u>Thm 2.5</u>: The space $H^0(R\Psi_{\mathrm{Spf}\, R_{\beta,m}} \overline{\mathbb{Q}}_\ell)$ is a fin dim cts repn of $W_{F_r} \times GL_n(\mathcal{O}/\varpi^m \mathcal{O})$ which vanishes outside $0 \leq i \leq n-1$

<u>Proof</u>: Compare with standard vanishing cycles for schemes + algebraize $\mathrm{Spf}\, R_{\beta,m}$ $\square$

Let $[R\Psi_\beta]$ be the elt of $\mathrm{Groth}(\underset{\text{cts}}{W_F} \times \underset{\text{smooth admiss}}{GL_n \mathcal{O}})$

induced by taking the alt sum of

$\varinjlim_m H^0(R^i\Psi_{\mathrm{Spf}\, R_{\beta,m}} \overline{\mathbb{Q}}_\ell)$

We apply this to $\phi_{\tau,h} \in C_c^\infty(GL_n(F_r))$ & get the associated function $f_{\tau,h} \in C_c^\infty(GL_n(F))$. Recall the main theorem (LLC):

<u>Thm 2.7</u>: (a) For any irred smooth rep π of $GL_n(F)$, $\exists!$ n-dim rep $\mathrm{rec}(\pi)$ of W_F s.t. $\forall \tau, h$ as before

$$\mathrm{tr}(f_{\tau,h} \mid \pi) = \mathrm{tr}(\tau \mid \mathrm{rec}(\pi))\, \mathrm{tr}(h \mid \pi)$$

Write $\sigma(\pi) := \mathrm{rec}(\pi)(\tfrac{1-n}{2})$ ← (twist by $(|\cdot| \circ \det)^{\frac{1-n}{2}}$)

(b) If π is a subquotient of the normalised parabolic induction of the irred. rep $\pi_1 \otimes \dots \otimes \pi_t$ of $GL_{n_1}(F) \times \dots \times GL_{n_t}(F)$ then $\sigma(\pi) =$

(c) The map $\pi \mapsto \sigma(\pi)$ induces a bij between iso classes of supercuspidal irred smooth rep of $GL_n(F)$ & the set of iso classes of irred n-dim reps of W_F.

(d) $\pi \mapsto \sigma(\pi)$ is compatible with twists, central chars, duals, L & ε-factors.

Recall from week 5, that for π a smooth irred rep of $GL_n(F)$ which is either ess. square integrable or generalised Speh repn, $\exists$ rep π_f of $G(\mathbb{A}_f)$, alg rep ξ of G s.t.

(unitary gp as before)

$$W_\xi^*(\pi_f) (:= \mathrm{Hom}_{G(\mathbb{A}_f)}(\pi_f, H_\xi^*)) \neq 0$$

(here $H_\xi^* = \varinjlim_K H^*(Sh_K, \mathcal{L}_\xi))$)

& the cmpt π_w of π_f is an unram twist of π w/p fixed.

Apply (†) to $\pi^\vee$ from thm 2.7, &

write $\pi_w = \pi^\vee \otimes \chi$.

& $\pi_{f,p} = \pi_w \otimes \pi_p^w \otimes \pi_{p,0}$ (get π_f) dual... ?

$$G(\mathbb{Q}_p) = GL_n(F) \times D'^\times \times \mathbb{Q}_p^\times$$

Let $\chi_{\pi_{p,0}}$ be the char of W_F assoc to $\pi_{p,0}$ by LCFT.

Then define $\mathrm{rec}(\pi) \overset{(*)}{:=} \frac{1}{a(\pi_f)} [W_\xi(\pi_f)^\vee] \otimes \chi_{\pi_{p,0}}^{-1} \otimes \chi^{-1}$.

this gives a $\mathbb{Q}$-lin combination of W_F-reps with +ve coeffts.

For any $\tau \in \mathrm{Frob}^r I_F \subset W_F$ & $h \in C_c^\infty(GL_n(\mathcal{O}))$

(taking vals in $\mathbb{Q}$), define

$$\phi_{\tau,h}(\beta) := \mathrm{tr}(\tau \times h^\vee \mid [R\Psi_\beta])$$

$\beta \in GL_n(F_r)$ here $h^\vee g := h((g^{-1})^t)$

<u>Thm 2.6</u>: This $\phi_{\tau,h}$ defines a function in $C_c^\infty(GL_n(F_r))$ with values in $\mathbb{Q}$ (indpt of ℓ.

~~Now we want to descend:~~

<u>§4 Descent:</u>

Goal: descend $\phi_{\tau,h}$ to a function $f_{\tau,h}$ on $GL_n(F)$.

ie $\phi_{\tau,h} = f_{\tau,h} \circ \mathrm{Norm}_{F_r/F}$.

Goes back to Arthur & Clozel .....

In general, say you have $\phi \in C_c^\infty(GL_n(F_r))$

$f \in C_c^\infty(GL_n(F))$.

<u>Define</u>:

"twisted orbital integral"

$$\Phi_{\phi,\sigma}(\delta) := \int_{GL_{n,\delta,\sigma}(F) \backslash GL_n(F_r)} \phi(g^{-1}\delta g^\sigma) \frac{dg_{F_r}}{dx}$$

$$\Phi_f(\gamma) := \int_{GL_{n,\gamma}(F)\backslash GL_n(F)} f(g^{-1}\gamma g)\frac{dg}{dt}$$

$\sigma = \mathrm{Frob}_{F_r/F}$.

$\delta \in GL_n F_r$

$\gamma = \mathrm{Norm}_{F_r/F}(\delta) \in GL_n(F)$

$GL_{n,\gamma}(F)$ = centralizer of γ in $GL_n F$

$GL_{n,\delta,\sigma}(F)$ = σ-centralizer of $\delta = \{g \in GL_n(F_r) : g^{-1}\delta g^{\sigma} = \delta\}$

We say that $\phi \in C_c^{\infty}(GL_n(F_r))$ & $f \in C_c^{\infty}(GL_n(F))$ are <u>associated</u> if they satisfy the following propn:

<u>Prop</u> (Arthur–Clozel)

(i) Assume $\phi \in C_c^{\infty}(GL_n(F_r))$. Then $\exists f \in C_c^{\infty}(GL_n(F))$ s.t. for regular $\gamma \in GL_n(F)$ we have

$$(*)\quad \Phi_f(\gamma) = \begin{cases} 0 & \text{if } \gamma \text{ is not a norm} \\ \Phi_{\phi,\sigma}(\delta) & \text{if } \gamma = \mathrm{Norm}_{F_r/F}(\delta),\ \delta \in GL_n(F_r) \end{cases}$$

(ii) conversely, given $f \in C_c^{\infty}(GL_n(F_r))$ satisfying $(*)$, $\exists \phi \in C_c^{\infty}(GL_n(F_r))$ s.t.

$\Phi_{\phi,\sigma}(\delta) = \Phi_f(\mathrm{Norm}_{F_r/F}(\delta))$ for $\delta \in GL_n(F_r)$.

s.t. $\forall \tau, h$ as above,

$$\mathrm{tr}\,(f_{\tau,h}\xi \mid \pi) = \mathrm{tr}(\tau \mid \mathrm{rec}(\pi))\,\mathrm{tr}(h \mid \pi)$$

1st step: (how to) embed it into a global situation.

Thm 10.2: Assume that π is an irred smooth rep of $GL_n F$ that is either essentially square integrable or a generalised Speh-repn. Then $\exists$ irred admiss π_f of $G(\mathbb{A}_f)$, & irred alg repn ξ of G s.t.

(i) $W^*_\xi(\pi_f) = \mathrm{Hom}_{G(\mathbb{A}_f)}(\pi_f, H^*_\xi) \neq 0$ for some $*$.

$$H^*_\xi = \varinjlim_K H^*(Sh_K, \mathcal{L}_{\xi,K})$$

(ii) the component π_w of π_f is an unram twist of π.

Formula for $\mathrm{rec}(\pi)$.

"it's not anything nice"

Kottwitz + Lemma 9.2

$\rightsquigarrow$

Corollary 10.1: Assume π_f is irred. admiss rep of $G(\mathbb{A}_f)$ with $W^*(\pi_f) \neq 0$.

Recall $G(\mathbb{Q}_p) = GL_n F \times D'^{\times} \times \mathbb{Q}_p^{\times}$.

Write $\pi_{f,p} = \pi_w \otimes \pi_p^w \otimes \pi_{p,0}$

Assume that $\pi_{p,0}$ is unram, & let $\chi_{\pi_{p,0}}$ be the char...

goal: prove ~~this~~ that $rec(\pi)$ is $\mathbb{Z}$-linear.

Firstly: ~~(*)~~ is to prove (*) gives a correct defn

$(tr(f_{\tau,h}|\pi) = tr(\tau|rec(\pi))\, tr(h|\pi)$

one uses

- global computations of Kottwitz

  computing $tr(f^{\vee}_{\tau,h} \times h \times e \times f^{p} \,|\, [H_{\xi}])$

  $G(\mathbb{A}_f) = GL_n(F) \times D'^{\times} \times \mathbb{Q}_p^{\times} \times (G(\mathbb{A}_f^{p}))$

  in terms of orbital integrals

($\leftrightarrow$ counting pts on Shimura varieties mod p

$\rightarrow$ integral models)

secondly: $rec(\pi)$ is actually $\mathbb{Z}$-lin combination of W_F-reps.

Cor 11.3: Let ρ an irred rep of $D'^{\times}$, let $\pi = JL(\rho)$. Suppose π is supercuspidal. Then as a virtual rep on $GL_n(\mathcal{O}) \times W_F$, the rep $[R\Psi_{\beta}](\rho^{\vee})$ is equal to $(-1)^{n-1}\pi^{\vee}|_{GL_n(\mathcal{O})^{\times}} \otimes rec(\pi)$

(in apt. Grothendieck group)

Proof: $tr(\tau|rec(\pi)).\, tr(h|\pi| ~~tr(f~~$

$= tr(f_{\tau,h}|\pi)$

$\underset{\text{base change}}{=} \operatorname{tr}\left((\phi_{\tau, h, \sigma}) \mid \pi \right)$

$= \dots$

$= (-1)^{n-1} \operatorname{tr}\left(\tau \times h^{\vee} \mid [R\psi^{\cdot}_{\beta}](\rho) \right)$