"Bijectivity"

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$$\begin{pmatrix} \text{irred} \\ \text{admissible} \\ \text{reps of } GL_n(K) \end{pmatrix} \longleftrightarrow \begin{pmatrix} F\text{-s.s.} \\ GL_n(\mathbb{C})\text{-reps} \\ \text{of the Weil-Deligne} \\ \text{group of } K \end{pmatrix}$$

$$\pi \longmapsto \sigma(\pi) := \mathrm{rec}(\pi)\left(\frac{1-n}{2}\right)$$

$$\mathrm{rec}(\pi) := \frac{1}{a(\pi_f)}[W_\xi(\pi_f)]^\vee \otimes \chi^{-1}_{\pi_{p,0}} \otimes \chi^{-1}$$

$$W^*_\xi(\pi_f) = \mathrm{Hom}_{G(\mathbb{A}_f)}(\pi_f, H^*_\xi)$$

$$H^*_\xi = \varinjlim_{\substack{G(\mathbb{A}_f) \supset K \\ \text{open} \\ \text{compact}}} H^*(Sh_K, \mathcal{F}_\xi)$$

(ξ an algebraic rep of G), chosen so that

$$[W_\xi(\pi_f)] \neq 0$$

$$\begin{tikzcd} \mathrm{Ga}(K) \arrow[r] & \hat{\mathbb{Z}} \\ W_K \arrow[u] \arrow[r] \arrow[ur, phantom, "L"] & \mathbb{Z} \arrow[u, hook] \end{tikzcd}$$

H^*_ξ is a rep of W_K but not continuous via $\overline{\mathbb{Q}_l} \cong \mathbb{C}$.

Theorem 12.1 • $\pi \longmapsto \sigma(\pi)$ satisfies

(1) For $n=1$, agrees with LCFT

(2) For $\pi = \pi_1 \boxplus \cdots \boxplus \pi_t$, where π_i is an irred rep of $GL_{n_i}(K)$,

$$\sigma(\pi) = \sigma(\pi_1) \oplus \cdots \oplus \sigma(\pi_t)$$

(3) $\sigma(\pi \otimes \chi \circ \det) = \sigma(\pi) \otimes \sigma(\chi)$, χ a character of K^*

(4) For K'/K cyclic extension of prime degree

$$\sigma(\pi)|_{W_{K'}} = \sigma(\Pi)$$

where Π is the base change lift to $GL_n(K')$.

(5) If $\sigma(\pi)$ is unramified, then π has a vector fixed by the Iwahori subgroup $\subseteq GL_n(\mathcal{O}_K)$

Proof (Sketch)

(1) Recall in §11, we saw the relation between $\mathrm{rec}(\pi)$ and the representation $\underbrace{[R\psi]}_{R^i\psi = 0 \text{ if } i<0,\, i>n-1}$.

So for $n=1$, just $R^0\psi$ can be nonzero.

For an element $\beta \in K^\times$, this defines a ϖ-divisible $\mathcal{O}$-module $\mathcal{F}$ over K, and $R\psi_\beta \longleftrightarrow$ looking at the $\mathrm{Gal}(K)$-action on $([\varpi^m]\mathcal{F})_m$

Coleman proved that this gives LCFT over K

$$\left\{\begin{matrix}\text{Fin ab quotients} \\ \text{of } K^\times\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Fin ab quotients} \\ \text{of } \mathrm{Gal}(\overline{K}/K)\end{matrix}\right\}.$$

(5) By (2), WLOG π is supercuspidal.

& by (1), $n \geq 2$.

Then Scholze shows $\sigma(\pi)^{I_K} = 0$,

by showing that $\mathrm{tr}(\mathrm{Frob}^r \sigma(\pi)^{I_K}) = 0 \quad \forall r \geq 0$. $\square$

Defn Such a family of maps

$$\left\{\begin{matrix}\text{irred smooth} \\ \text{reps of } GL_n(F)\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}n\text{-dim} \\ \text{reps of} \\ W_F\end{matrix}\right\}$$

for all $F/\mathbb{Q}_p$ finite, satisfying (1)-(5) is called a functorial extension of LCFT

③ <u>Thm 12·3</u> For any functorial extension of LCFT,

$$\left\{\begin{matrix}\text{irred supercusp}\\ \text{reps of } GL_n(F)\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}n\text{-dim}\\ \text{reps of } W_F\end{matrix}\right\}$$

is <u>injective</u> with image consisting exactly the irreducible n-dim reps.

<u>Proof</u>: Induction on n.

<u>$n=1$</u> ✓ (LCFT).

$\exists\, F = F_0 \subset F_1 \subset \dots \subset F_m$ of cyclic Galois extensions of prime degree, such that base change lift of π to F_{m-1} is irreducible supercuspidal, but lift to but the lift ~~of~~ to F_m isn't. Replace F by F_{m-1}.

<u>Result in Arthur–Clozel</u> ↑ Π = base change lift of π ~~[illegible]~~ to F_1.

$$\Rightarrow \Pi \cong \Pi_1 \boxplus \Pi_1^{\tau} \boxplus \dots \boxplus \Pi_1^{\tau^{g-1}}$$

with $\Pi_1 \not\cong \Pi_1^{\tau}$. (Here $Gal(F_1/F) \cong \mathbb{Z}/g\mathbb{Z}$)

& Π_1 is an irreducible supercuspidal rep of $GL_{n/g}(F_1)$

By induction,

$\sigma(\Pi_1)$ is irreducible and $\sigma(\Pi_1^{\tau}) \not\cong \sigma(\Pi_1)$.

Now $\sigma(\pi)|_{W_{F_1}} \cong \sigma(\Pi_1) \oplus \dots \oplus \sigma(\Pi_1)^{\tau^{g-1}}$

with $\sigma(\Pi_1) \not\cong \sigma(\Pi_1)^{\tau}$ by induction.

Hence $\sigma(\pi)$ is irreducible (Clifford theory).

<u>Now want to show bijectivity</u>

Start with an irred n-dim rep of W_F.

Take $F_1 \subset \dots \subset F_m$ cyclic s.t. $\sigma|_{W_{F_{m-1}}}$ is irreducible, but $\sigma|_{W_{F_m}}$ isn't.

(4) Then $\exists\, \Sigma$, an irred n/g-dim rep of W_{F_m}, and a character τ s.t.

$$\sigma|_{W_{F_m}} \cong \Sigma \oplus \Sigma^{\tau} \oplus \cdots \oplus \Sigma^{\tau^{g-1}}.$$

Then by induction $\exists!\, \Pi$ supercuspidal rep of $GL_{n/g}(F_m)$ s.t.

$$\sigma|_{F_m} \cong \mathrm{Ind}_{W_{F_m}}^{W_{F_{m-1}}} \Sigma.$$

$\longleftrightarrow$ have Π_{m-1}, an irred rep of $GL_n(F_{m-1})$.

(~~[illegible]~~ which lifts $\Pi_m = \Pi \boxplus \cdots \boxplus \Pi^{\tau^{g-1}}$).

Remains to show: if σ is an irreducible rep of W_F, F_1/F cyclic, and $\sigma|_{W_{F_1}}$ is irred and corresponds to a unique supercuspidal π_1 of $GL_n(F_1)$, then $\exists!\, \pi$ of $GL_n(F)$ corresponding to σ.

$$\sigma|_{W_{F_1}}^{\tau} \cong \sigma|_{W_{F_1}} \Rightarrow \pi_1 \cong \pi_1^{\tau}$$

Arthur–Clozel show that this means that π_1 lifts to a supercuspidal π on $GL_n(F)$.

(i.e. π_1 is the base change lift of π).

So we have π and $\sigma(\pi)$ s.t.

$$\sigma(\pi)|_{W_{F_1}} \cong \sigma|_{W_{F_1}}.$$

Also $\sigma(\pi) = \sigma \otimes \chi$ for χ a character of $Gal(F_1/F)$ (Mackey formula?)

5 Hence $\sigma(\pi \otimes \chi^{-1}\circ\det) = \sigma(\pi) \otimes \sigma(\chi)^{-1} = \sigma$. $\square$

Tiny bit about § 14

In § 14, Scholze shows compatibility with L and ε factors of pairs.
By Henniart's "Numerical Local Langlands" this, together with other compatibilities Thm 14.1 uniquely determines $\pi \longmapsto \sigma(\pi)$.
In particular, it's the same as Harris-Taylor.