MOD p NON-ABELIAN LUBIN-TATE THEORY FOR $\mathrm{GL}_2(\mathbb{Q}_p)$

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Abstract. We analyse the mod p étale cohomology of the Lubin-Tate tower both with compact support and without support. We prove that there are no supersingular representations in the H^1_c of the Lubin-Tate tower. On the other hand, we show that in H^1 of the Lubin-Tate tower appears the mod p local Langlands correspondence and the mod p local Jacquet-Langlands correspondence, which we define in the text. We show also a similar result in the p-adic setting. We discuss the local-global compatibility part of the Buzzard-Diamond-Jarvis conjecture which appears naturally in this context.

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1. Introduction

In recent years, the mod p and p-adic local Langlands correspondences emerged as a form of refinement of the l-adic Langlands correspondence for $l = p$. This program was basically started by Christophe Breuil and then, by the work of many people the p-adic Local Langlands correspondence was established for $\mathrm{GL}_2(\mathbb{Q}_p)$ (see [Co] for a final step). Unfortunately, as for now, it is hard to predict how the conjectures should look like for $\mathrm{GL}_2(F)$ where F is a finite extension of $\mathbb{Q}_p$ (or other groups) basically because there are too many objects on the automorphic side (see [BP]), so that the pure representation theory does not indicate which representations of GL_2 should we choose for our correspondence. A possible remedy to this situation might come by looking at the mod p and completed p-adic cohomology of the Lubin-Tate tower. Let us remind the reader that the classical Local Langlands correspondence was also firstly proved for GL_2 by representation-theoretic methods. Only afterwards by using geometric arguments and finding the correspondence in the l-adic cohomology of the Lubin-Tate tower, it was proved for GL_n by Harris and Taylor in [HT]. Our aim is to do a step in this direction in the $l = p$ setting, hoping that this will give us an insight how to define a correspondence for other groups than $\mathrm{GL}_2(\mathbb{Q}_p)$.

By the recent work of Emerton (see [Em2]) we know that the p-adic completed (resp. mod p) cohomology of the tower of modular curves realizes the p-adic (resp. mod p) Local Langlands correspondence. In this article we will obtain an analogous result for the mod p cohomology of the Lubin-Tate tower over $\mathbb{Q}_p$. In fact, we will analyse both the cohomology with compact support and the cohomology without support of the Lubin-Tate tower. There are two main results which we prove:

(1) In the first cohomology group $H^1_{LT,\bar{\mathbb{F}}_p}$ of the Lubin-Tate tower appears the mod p local Langlands correspondence and the mod p Jacquet-Langlands correspondence meaning that there is an injection of representations

$$\sigma \otimes \pi \otimes \bar{\rho} \hookrightarrow H^1_{LT,\bar{\mathbb{F}}_p}$$

where π is a supersingular representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, $\bar{\rho}$ is its associated local mod p Galois representation and σ is the naive mod p Jacquet-Langlands correspondence (for details, see Section 8).

(2) The first cohomology group $H^1_{LT,c,\bar{\mathbb{F}}_p}$ with compact support of the Lubin-Tate tower does not contain any supersingular representations. This suprising result shows that the mod p situation is much different from its mod l analogue. It also permits us to show that in H^1 of the ordinary locus appears the mod p local Langlands correspondence - again a fact which is different from the l-adic setting.

Before sketching how we obtain the above results, let us outline what is a difference with the non-abelian Lubin-Tate theory in the l-adic case. When $l \neq p$ the comparison between the Lubin-Tate tower and the modular curve tower is made via vanishing cycles. For that, we need to know that the stalks of vanishing cycles gives the cohomology of the Lubin-Tate tower, or in other words we need an analogue of the theorem proved by Berkovich in [Ber3]. But when $l = p$, the statement does not hold anymore (see Remark 3.8.(iv) in [Ber3]) and hence we cannot imitate directly the arguments from the l-adic theory.

To circumvent this difficulty, we work from the beginning at the rigid-analytic level and consider embeddings from the ordinary and the supersingular tubes into modular curves. This gives two long exact sequences of cohomology, depending on whether we take a compact support or a support in the ordinary locus and we start our analysis by resuming neccessary facts about the geometry of modular curves. We show a decomposition of the ordinary locus, which proves that its cohomology is induced from some proper parabolic subgroup of GL_2. We use this fact several times in order to force vanishing of the cohomology of ordinary locus after localising at a supersingular representation π of $\mathrm{GL}_2(\mathbb{Q}_p)$. We then recall standard facts about admissible representations and review the functor of localisation at π which comes out of the work of Paskunas.

We then turn to the analysis of the supersingular locus. In this context, naturally appears a quaternion algebra $D^\times/\mathbb{Q}$ which is ramified exactly at p and ∞. We define the local fundamental representation of Deligne in our setting (which appeared for the first time in the letter of Deligne [De]) and we show a decomposition of the cohomology of supersingular locus. At this point we will be able to show that H^1 of the tower of modular curves injects into H^1 of the Lubin-Tate tower hence proving part of (1).

Having established this result, we start analysing mod p representations of the p-adic quaternion algebra and define a candidate for the mod p Jacquet-Langlands correspondence $\sigma_\mathfrak{m}$ which we later show to appear in the cohomology. It will a priori depend on a global input, namely a maximal ideal $\mathfrak{m}$ of a Hecke algebra corresponding to some modular mod p Galois representation $\bar{\rho}$, but we conjecture that it is independent of $\mathfrak{m}$. This is reasonable as it would follow from the local-global compatibility part of the Buzzard-Diamond-Jarvis conjecture. After further analysis of $\sigma_\mathfrak{m}$ we are able to finish the proof of (1).

Using similar techniques, we start analysing the cohomology with compact support of the Lubin-Tate tower. By using the Hochschild-Serre spectral sequence, we are able to reduce (2) to the question of whether supersingular Hecke modules of the mod p Hecke algebra at the pro-p Iwahori level appear in the H^1_c of the Lubin-Tate tower at the pro-p Iwahori level. We solve this question by explicitely computing some cohomology groups.

While proving the above theorems, we will also prove that the first cohomology group of the Lubin-Tate tower and the first cohomology group of the ordinary locus are non-admissible smooth representations. To obtain it, we are basically using the fact that if the cohomology of the ordinary locus is admissible then its localisation at π vanishes. For the Lubin-Tate tower, we have to use additionally a fact that a mod p Jacquet-Langlands correspondence $\sigma_\mathfrak{m}$ is a representation of $D^\times(\mathbb{Q}_p)$ of infinite length. This indicates that already for $D^\times(\mathbb{Q}_p)$ the mod p Langlands correspondence is complicated (as in the work of [BP], representations in question are not of finite length). On the other hand, the case of $D^\times(\mathbb{Q}_p)$ is much simpler than that of $\mathrm{GL}_2(F)$ for F a finite extension of $\mathbb{Q}_p$, and hence we might be able to describe $\sigma_\mathfrak{m}$ precisely. Natural question in this discussion is the local-global compatibility part of the Buzzard-Diamond-Jarvis conjecture (see Conjecture 4.7 in [BDJ]) which says that we have an isomorphism

$$\mathbf{F}[\mathfrak{m}] \simeq \sigma_\mathfrak{m} \otimes \pi^p(\bar{\rho})$$

where $\mathbf{F}$ denotes locally constant functions on $D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)$ with values in $\bar{\mathbb{F}}_p$ and $\pi^p(\bar{\rho})$ is a representation of $\mathrm{GL}_2(\mathbb{A}_f^p)$ associated to $\bar{\rho}$ by the modified Langlands correspondence.

At the end, we remark that our arguments work well in the $l \neq p$ setting and bypasses the use of vanishing cycles. As some of our arguments are geometric, we can also get similar results in the p-adic setting which we mention in the text. Also, the geometry of modular curves is very similar to the geometry of Shimura curves and hence we hope that some of the reasonings in this article will give an insight into the nature of the mod p local Langlands correspondence of $\mathrm{GL}_2(F)$ for F a finite extension of $\mathbb{Q}_p$.

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2. Geometry of modular curves

Let $X(Np^m)$ be the Katz-Mazur compactification of the modular curve associated to the moduli problem $(\Gamma(p^n), \Gamma_1(N))$ (see [KM]) which is defined over $\mathbb{Z}[1/N, \zeta_{p^n}]$, where ζ_{p^n} is a primitive p^n-th root of unity, that is $X(Np^m)$ parametrizes (up to isomorphism) triples (E, ϕ, α), where E is an elliptic curve, $\phi : (\mathbb{Z}/p^n\mathbb{Z})^2 \to E[p^n]$ is a Drinfeld level structure and $\alpha : \mathbb{Z}/N\mathbb{Z} \to E[N]$ is a $\Gamma_1(N)$-structure. We consider the integral model of it defined over $\mathbb{Z}_p^{nr}[\zeta_{p^n}]$, where $\mathbb{Z}_p^{nr}$ is the maximal unramified extension of $\mathbb{Z}_p$, which we will denote also by $X(Np^m)$. Let us denote by $X(Np^m)^{an}$ the analytification of $X(Np^m)$ which is a Berkovich space.

Recall that there exists a reduction map $\pi : X(Np^m)^{an} \to \overline{X(Np^m)}$, where $\overline{X(Np^m)}$ is the special fiber of $X(Np^m)$. We define $\overline{X(Np^m)_{ss}}$ (resp. $\overline{X(Np^m)_{ord}}$) to be the set of supersingular (resp. ordinary) points in $\overline{X(Np^m)}$. Define the tubes $X(Np^m)_{ss} = \pi^{-1}(\overline{X(Np^m)_{ss}})$ and $X(Np^m)_{ord} = \pi^{-1}(\overline{X(Np^m)_{ord}})$ inside $X(Np^m)^{an}$ of supersingular and ordinary points respectively.

2.1. **Two exact sequences.** We know that $X(Np^m)_{ss}$ is an open analytic subspace of $X(Np^m)^{an}$ isomorphic to some copies of Lubin-Tate spaces, where number of copies is equal to the number of points in $\overline{X(Np^m)_{ss}}$ (see section 3 of [Bu]). We have a decomposition $X(Np^m)^{an} = X(Np^m)_{ss} \cup X(Np^m)_{ord}$ and we put $j : X(Np^m)_{ss} \hookrightarrow X(Np^m)^{an}$ and $i : X(Np^m)_{ord} \to X(Np^m)^{an}$. Let F be a sheaf in the étale topoi of $X(Np^m)^{an}$. By the general formalism of six operations (due in this setting to Berkovich, see [Ber5]) we have a short exact sequence:

$$0 \to j_! j^* F \to F \to i_* i^* F \to 0$$

which gives a long exact sequence of étale cohomology groups:

$$... \to H^0(X(Np^m)_{ord}, i^*F) \to H^1_c(X(Np^m)_{ss}, F) \to H^1(X(Np^m)^{an}, F) \to H^1(X(Np^m)_{ord}, i^*F) \to ...$$

We define:

$$\mathrm{Ker}(X(Np^m), i^*F) = im(H^0(X(Np^m)_{ord}, i^*F) \to H^1_c(X(Np^m)_{ss}, F))$$

and

$$\mathrm{Im}(X(Np^m), i^*F) = im(H^1(X(Np^m)^{an}, F) \to H^1(X(Np^m)_{ord}, i^*F))$$

so that we have an exact sequence:

$$0 \to \mathrm{Ker}(X(Np^m), i^*F) \to H^1_c(X(Np^m)_{ss}, F) \to H^1(X(Np^m)^{an}, F) \to \mathrm{Im}(X(Np^m), i^*F) \to 0$$

which we will analyse in the following.

On the other hand, we can consider a similar exact sequence for the cohomology without compact support, but instead considering support on the ordinary locus. This results in the long exact sequence

$$... \to H^1_{X_{ord}}(X(Np^m)^{an}, F) \to H^1(X(Np^m)^{an}, F) \to H^1(X(Np^m)_{ss}, i^*F) \to H^2_{X_{ord}}(X(Np^m)^{an}, F) \to ...$$

where we have denoted by $H^1_{X_{ord}}(X(Np^m)^{an}, F)$ the étale cohomology of $X(Np^m)^{an}$ with support on $X(Np^m)_{ord}$. Because of the vanishing of the cohomology with compact support of the supersingular locus localised at π (see the explanation in the next sections), this exact sequence will be of more importance to us later on. In our article we analyse those two exact sequences simultanously.

2.2. **Decomposition of ordinary locus.** Let us recall that we have the Weil pairing on elliptic curves

$$e_{p^m} : E[p^m] \times E[p^m] \to \mu_{p^m}$$

Denote by ζ_{p^m} a p^m-th primitive root of unity and recall that the set of irreducible components of $\overline{X(Np^m)}$ consists of smooth curves $C_{a,b}(Np^m)$ defined on points by:

$$C_{a,b}(Np^m)(S) = \{(E, \phi, \alpha) \in \overline{X(Np^m)}(S) \mid e_{p^m}(\phi(\tbinom{1}{0}), \phi(\tbinom{0}{1})) = \zeta^a_{p^m} \text{ and } \mathrm{Ker}\,\phi = b\}$$

where $a \in (\mathbb{Z}/p^m\mathbb{Z})^*$ and $b \in \mathbb{P}^1(\mathbb{Z}/p^m\mathbb{Z})$ is regarded as a line in $\mathbb{Z}/p^m\mathbb{Z} \times \mathbb{Z}/p^m\mathbb{Z}$.

For an ordinary rigid analytic elliptic curve E, take $K(E)$ to be $ker(\pi : E \to \overline{E})$, the kernel of the reduction map. After the theory of canonical subgroups (see section 3 of [Bu]), there exists a canonical subgroup $K_m(E) \subset K(E)$ which is cyclic and of order p^m. We introduce subspaces $\mathbb{X}_{a,b}(Np^m)$ defined on points by:

$$\mathbb{X}_{a,b}(Np^m)(S) = \{(E,\phi,\alpha) \in X(Np^m)_{ord}(S) \mid e_{p^m}(\phi(\begin{smallmatrix}1\\0\end{smallmatrix}), \phi(\begin{smallmatrix}0\\1\end{smallmatrix})) = \zeta_{p^m}^a \text{ and } \phi^{-1}(K_m(E)) = b\}$$

The reduction of each $\mathbb{X}_{a,b}(Np^m)$ is precisely $C_{a,b}(Np^m)$ and hence $\{\mathbb{X}_{a,b}(Np^m)\}$ for $a \in (\mathbb{Z}/p^m\mathbb{Z})^*$, $b \in \mathbb{P}^1(\mathbb{Z}/p^m\mathbb{Z})$ form a decomposition of the ordinary locus $X(Np^m)_{ord}$ because different $C_{a,b}(Np^m)$ intersect only at supersingular points. The spaces $\mathbb{X}_{a,b}(Np^m)$ may be regarded as analytifications of Igusa curves. For a detailed discussion, see [Col].

There is an action of $\mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z})$ on $X(Np^m)^{an}$ which is given on points by:

$$(E,\phi,\alpha)\cdot g = (E, \phi\circ g, \alpha)$$

for $g \in \mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z})$. Observe that if $e_{p^m}(\phi(\begin{smallmatrix}1\\0\end{smallmatrix}), \phi(\begin{smallmatrix}0\\1\end{smallmatrix})) = \zeta_{p^m}^a$, then $e_{p^m}((\phi\circ g)(\begin{smallmatrix}1\\0\end{smallmatrix}), (\phi\circ g)(\begin{smallmatrix}0\\1\end{smallmatrix})) = \zeta_{p^m}^{a\cdot \det g}$ for $g \in \mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z})$ and so g induces an isomorphism between $\mathbb{X}_{a,b}(Np^m)$ and $\mathbb{X}_{a\cdot\det g, g^{-1}\cdot b}(Np^m)$.

For $b \in \mathbb{P}^1(\mathbb{Z}/p^m\mathbb{Z})$ there is a Borel subgroup $B_m(b)$ in $\mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z})$ which fixes b and hence the Borel subgroup $B_m(b)^+ = B_m(b) \cap \mathrm{SL}_2(\mathbb{Z}/p^m\mathbb{Z})$ in $\mathrm{SL}_2(\mathbb{Z}/p^m\mathbb{Z})$ stabilises $\mathbb{X}_{a,b}(Np^m)$.

Let $b = \infty = (\begin{smallmatrix}1\\0\end{smallmatrix}) \in \mathbb{P}^1(\mathbb{Z}/p^m\mathbb{Z})$. By the above considerations we have

$$H^i(X(Np^m)_{ord}, i^*F) = \bigoplus_{a,b} H^i(\mathbb{X}_{a,b}(Np^m), (i^*F)_{|\mathbb{X}_{a,b}(Np^m)}) \simeq \mathrm{Ind}_{B_m(\infty)}^{\mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z})}\left(\bigoplus_a H^i(\mathbb{X}_{a,\infty}(Np^m), (i^*F)_{|\mathbb{X}_{a,\infty}})\right)$$

and also

$$H^1_{X_{ord}}(X(Np^m)^{an}, F) = \mathrm{Ind}_{B_m(\infty)}^{\mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z})}\left(\bigoplus_a H^1_{\mathbb{X}_{a,\infty}(Np^m)}(X(Np^m)^{an}, F)\right)$$

This also gives a similar induction for $\mathrm{Ker}(X(Np^m), i^*F)$ and $\mathrm{Im}(X(Np^m), i^*F)$ introduced in 2.1, but this time we have only an injection (respectively, a surjection) from the induced comhomology groups:

$$\mathrm{Ker}(X(Np^m), i^*F) \twoheadleftarrow \mathrm{Ind}_{B_m(\infty)}^{\mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z})}\left(\bigoplus_a im(H^0(\mathbb{X}_{a,\infty}(Np^m), (i^*F)_{|\mathbb{X}_{a,\infty}(Np^m)}) \to H^1_c(X(Np^m)_{ss}, F))\right)$$

$$\mathrm{Im}(X(Np^m), i^*F) \subset \mathrm{Ind}_{B_m(\infty)}^{\mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z})}\left(\bigoplus_a im(H^1(X(Np^m)^{an}, F) \to H^1(\mathbb{X}_{a,\infty}(Np^m), (i^*F)_{|\mathbb{X}_{a,\infty}(Np^m)}))\right)$$

Those results will be extremely useful for us later on, when we introduce a localisation at a given supersingular representation.

2.3. **Supersingular points.** Let us denote by D the quaternion algebra which is ramified precisely at p and at ∞. We recall the description of supersingular points $\overline{X(Np^m)_{ss}}$ which has appeared in [De] and then was explained in [Ca], sections 9.4 and 10.4. Fix a supersingular elliptic curve $\overline{E}$ over $\mathbb{F}_p$ and a two-dimensional vector space V over $\mathbb{Q}_p$. Let $\det(\overline{E})) = \mathbb{Z}$ be the determinant of $\overline{E}$. Denote by $W(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ the Weil group of $\mathbb{F}_p$ and put

$$\Delta = \left(W(\bar{\mathbb{F}}_p/\mathbb{F}_p) \times \mathrm{Isom}(\det(\overline{E}) \otimes_{\mathbb{Z}} \mathbb{Q}_p(1), \wedge^2 V)\right)/\sim$$

where $\sim$ is defined by $(\sigma,\beta) \sim (\sigma\,\mathrm{Frob}^k, p^{-k}\beta)$ for $k \in \mathbb{Z}$, where $\mathrm{Frob} : x \mapsto x^p$ is a Frobenius map. We define K_m to be the kernel of $D^\times(\mathbb{Z}_p) \to D^\times(\mathbb{Z}/p^m\mathbb{Z})$ and we let $K(N) = \{(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}) \in \mathrm{GL}_2(\mathbb{Z}) \mid a \equiv 1 \mod N \text{ and } c \equiv 0 \mod N\}$, viewed as a subgroup of $\mathrm{GL}_2(\mathbb{A}_f^p)$ by the diagonal embedding. Then:

$$\overline{X(Np^m)_{ss}} = \Delta/K_m \times_{D^\times(\mathbb{Q})} \mathrm{GL}_2(\mathbb{A}_f^p)/K(N)$$

Every $\delta \in \Delta/K_m$ furnishes a supersingular elliptic curve $E(\delta)$ with a $(\Gamma(p^m), \Gamma_1(N))$-structure, so that for every $\delta \in \Delta$ we can consider the Lubin-Tate tower $LT_\delta = \varprojlim_m LT_\delta(p^m)$, which is the generic fiber of the deformation space of the formal group attached to $E(\delta)$ (see [Da1] for details on the Lubin-Tate tower). Let us denote by $\mathbb{E}(\delta)$ the universal formal group deforming the formal group attached to

$E(\delta)$ and let $\mathbb{E}(\Delta) = \coprod_{\delta\in\Delta}\mathbb{E}(\delta)$. By 9.4 of [Ca], the universal formal group over $\varprojlim_{N,p^m}\overline{X(Np^m)_{ss}}$ is isomorphic to $\mathbb{E}(\Delta)\times_{D^\times(\mathbb{Q})}\mathrm{GL}_2(\mathbb{A}_f^p)$ and hence we conclude that

$$\varprojlim_{Np^m} X(Np^m)_{ss} \simeq LT_\Delta \times_{D^\times(\mathbb{Q})} \mathrm{GL}_2(\mathbb{A}_f^p)$$

where $LT_\Delta = \coprod_{\delta\in\Delta} LT_\delta$. Taking $K_mK(N)$-invariants we also get

$$X(Np^m)_{ss} \simeq LT_{\Delta/K_m} \times_{D^\times(\mathbb{Q})} \mathrm{GL}_2(\mathbb{A}_f^p)/K(N)$$

where $LT_{\Delta/K_m} = \coprod_{\delta\in\Delta/K_m} LT_\delta(p^m)$.

These results will allow us later on to define the local fundamental representation and analyze the action of the quaternion algebra $D^\times$.

3. Admissibility of cohomology groups

In this section we will recall the notion of admissibility in the context of mod p and p-adic representations. It will be crucial in our study of cohomology.

3.1. **General facts and definitions.** We start with general facts about admissible representations. In our definitions, we will follow [Em3]. Let K be a finite extension of $\mathbb{Q}$ with ring of integers $\mathcal{O}$, uniformiser ϖ and residue field k. Let $C(\mathcal{O})$ denote the category of complete Noetherian local $\mathcal{O}$-algebras having finite residue fields. Let us consider $A \in C(\mathcal{O})$.

Definition 3.1. *Let V be a representation of G over A. A vector $v \in V$ is smooth if v is fixed by some open subgroup of G and v is annihilated by some power $\mathfrak{m}^i$ of the maximal ideal of A. Let V_{sm} denote the subset of smooth vectors of V. We say that a G-representation V over A is smooth if $V = V_{sm}$.*

A smooth G-representation V over A is admissible if $V^H[\mathfrak{m}^i]$ (the $\mathfrak{m}^i$-torsion part of the subspace of H-fixed vectors in V) is finitely generated over A for every open compact subgroup H of G and every $i \geq 0$.

Definition 3.2. *We say that a G-representation V over A is ϖ-adically continuous if V is ϖ-adically separated and complete, $V[\varpi^\infty]$ is of bounded exponent, V/ϖ^iV is a smooth G-representation for any $i \geq 0$.*

Definition 3.3. *A ϖ-adically admissible representation of G over A is a ϖ-adically continuous representation V of G over A such that the induced G-representation on $(V/\varpi V)[\mathfrak{m}]$ is admissible smooth over $A/\mathfrak{m}$.*

Remark 3.4. *This definition implies that for every $i \geq 0$, the G-representation V/ϖ^iV is smooth admissible. See Remark 2.4.8 in* [Em3].

Remark 3.5. *The ϖ-adically admissible representations are called Banach admissible in* [ST]*, where authors work over a field. We will use this latter notion most of the time even in the context of rings.*

Proposition 3.6. *The categories of admissible K-Banach representations and of admissible F-representations are abelian, where K is a finite extension of $\mathbb{Q}_p$ and F is a finite field.*

Proof. Both categories are (anti-)equivalent to categories of finitely generated augmented modules over certain completed group rings. See Proposition 2.2.13 and 2.4.11 in [Em3]. □

Now, we will prove an analogue of Lemma 13.2.3 from [Bo] in the $l = p$ setting. We will later apply this lemma to the cohomology of the ordinary locus to force its vanishing after localisation at a supersingular representation of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Lemma 3.7. *For any smooth admissible representation (π, V) of the parabolic subgroup $P \subset G$ over A, the unipotent radical U of P acts trivially on V.*

Proof. Let L be a Levi subgroup of P, so that $P = LU$. Let $v \in V$ and let $K_P = K_L K_U$ be a compact open subgroup of P such that $v \in V^{K_P}$. We choose an element z in the centre of L such that:

$$z^{-n}K_P z^n \subset ... \subset z^{-1}K_P z \subset K_P \subset zK_P z^{-1} \subset ... \subset z^n K_P z^{-n} \subset ...$$

and $\bigcup_{n\geq 0} z^n K_P z^{-n} = K_L U$. For every n and m, modules $V^{z^{-n}K_P z^n}[\mathfrak{m}^i]$ and $V^{z^{-m}K_P z^m}[\mathfrak{m}^i]$ are of the same length for every $i \geq 0$, as they are isomorphic via $\pi(z^{n-m})$ and hence we have not only an isomorphism but an equality $V^{z^{-n}K_P z^n}[\mathfrak{m}^i] = V^{z^{-m}K_P z^m}[\mathfrak{m}^i]$. By smoothness, for every $x \in V$, there exists i such that $x \in V[\mathfrak{m}^i]$. Thus we have $x \in V^{K_P}[\mathfrak{m}^i] = V^{z^{-n}K_P z^n}[\mathfrak{m}^i] = V^{K_L U}[\mathfrak{m}^i]$ which is contained in $V^U[\mathfrak{m}^i]$. □

Lemma 3.8. *For any ϖ-adically admissible representation (π, V) of the parabolic subgroup $P \subset G$ over A, the unipotent radical U of P acts trivially on V.*

Proof. By the remark above, each $V/\varpi^i V$ is admissible, and hence the preceding lemma applies, so that U acts trivially on each $V/\varpi^i V$. But $V = \varprojlim_i V/\varpi^i V$, hence U acts trivially on V. □

3.2. **Completed cohomology and admissibility.** In [Em1], Emerton has introduced the completed cohomology, which plays a crucial role in the p-adic Langlands program. The most important thing for us right now is the fact that those cohomology groups for modular curves are admissible as $\mathrm{GL}_2(\mathbb{Q}_p)$-representations. We have

Proposition 3.9. *The following $\mathrm{GL}_2(\mathbb{Q}_p)$-representations are admissible:*

a) $\widehat{H}^1(X(N), K) = \varprojlim_s \varinjlim_m H^1(X(Np^m)^{an}, \mathcal{O}_K/\varpi^s\mathcal{O}_K) \otimes_{\mathcal{O}_K} K$

b) $\widehat{H}^1(X(N), \bar{\mathbb{F}}_p) = \varinjlim_m H^1(X(Np^m)^{an}, \bar{\mathbb{F}}_p)$

Proof. This is Theorem 2.1.5 of [Em1] (see also Theorem 1.16 in [CE]). □

By formal properties of the category of admissible representations, which form a Serre subcategory of the category of smooth representations, the above result permits us to deduce admissibility for other cohomology groups which are of interest to us. Let us only remark that we can define also the completed cohomology with compact support, at least for the Lubin-Tate tower:

Remark 3.10. *A priori, cohomology with compact support is a covariant functor. But using the adjunction map*

$$\Lambda \to \pi_*\pi^*\Lambda \simeq \pi_!\pi^!\Lambda$$

where Λ is a constant sheaf and $\pi : X(Np^{m+1})_{ss} \to X(Np^m)_{ss}$ is finite (hence $\pi_* = \pi_!$) and étale (hence $\pi^! = \pi^*$) by the properties of Lubin-Tate tower. Thus we get maps $H^i_c(X(Np^m)_{ss}, \Lambda) \to H^i_c(X(Np^{m+1})_{ss}, \Lambda)$ compatible with $H^i(X(Np^m)^{an}, \Lambda) \to H^i(X(Np^{m+1})_{ss}, \Lambda)$.

We start firstly by analysing cohomology groups which appear in the exact sequence for the cohomology with compact support. From the above proposition we have

Proposition 3.11. *The following $\mathrm{GL}_2(\mathbb{Q}_p)$-representations are admissible:*

a) $\widehat{\mathrm{Im}}(N, K) = \varprojlim_s \varinjlim_m \mathrm{Im}(X(Np^m), \mathcal{O}_K/\varpi^s\mathcal{O}_K) \otimes_{\mathcal{O}_K} K$

b) $\widehat{\mathrm{Im}}(N, \bar{\mathbb{F}}_p) = \varinjlim_m \mathrm{Im}(X(Np^m), \bar{\mathbb{F}}_p)$

Proof. By Proposition 2.2.13 in [Em3], admissible representations form a Serre subcategory of the abelian category of smooth representations, hence if $V \twoheadrightarrow W$ is a surjection of smooth representations, where V is admissible, then also W is admissible. Hence we conclude by the proposition above and the fact that considered representations are smooth (we use the fact that a quotient of a smooth representation is smooth to get smoothness of $\widehat{\mathrm{Im}}(N, -)$). □

Proposition 3.12. *The following $\mathrm{GL}_2(\mathbb{Q}_p)$-representations are admissible:*

a) $\widehat{\mathrm{Ker}}(N, K) = \varprojlim_s \varinjlim_m \mathrm{Ker}(X(Np^m), \mathcal{O}_K/\varpi^s\mathcal{O}_K) \otimes_{\mathcal{O}_K} K$

b) $\widehat{\mathrm{Ker}}(N, \bar{\mathbb{F}}_p) = \varinjlim_m \mathrm{Ker}(X(Np^m), \bar{\mathbb{F}}_p)$

Proof. The number of connected components of $X(Np^m)_{ord}$ is finite and let $d(Np^m)$ be their number. For $s > 0$, we have

$$H^0(X(Np^m)_{ord}, \mathcal{O}_K/\varpi^s\mathcal{O}_K) = (\mathcal{O}_K/\varpi^s\mathcal{O}_K)^{d(Np^m)}$$

hence $\varprojlim_s \varinjlim_m H^0(X(Np^m)_{ord}, \mathcal{O}_K/\varpi^s\mathcal{O}_K)$ is admissible as well as $\varinjlim_m H^0(X(Np^m)_{ord}, \bar{\mathbb{F}}_p)$. From the fact that admissible representations form a Serre subcategory of smooth representations we conclude. □

We deduce finally

Proposition 3.13. *The following* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representations are admissible:*
a) $\widehat{H}^1_c(X(N)_{ss}, \mathcal{O}_K/\varpi^s\mathcal{O}_K) = \varinjlim_m H^1_c(X(Np^m)_{ss}, \mathcal{O}_K/\varpi^s\mathcal{O}_K)$
b) $\widehat{H}^1_c(X(N)_{ss}, \bar{\mathbb{F}}_p) = \varinjlim_m H^1_c(X(Np^m)_{ss}, \bar{\mathbb{F}}_p)$

Proof. Let A be either $\mathcal{O}_K/\varpi^s\mathcal{O}_K$ or $\bar{\mathbb{F}}_p$. We consider the exact sequence arising from 1.1:

$$0 \to \widehat{\mathrm{Ker}}(N, A) \to \widehat{H}^1_c(X(N)_{ss}, A) \to \widehat{H}^1(X(N), A) \to \widehat{\mathrm{Im}}(N, A) \to 0$$

and we conclude using again the fact that admissible representations form a Serre subcategory of smooth representations and the propositions proved above. □

Let us also define for a future use

$$\widehat{H}^1_c(X(N)_{ss}, K) = \varprojlim_s \varinjlim_m H^1_c(X(Np^m)_{ss}, \mathcal{O}_K/\varpi^s\mathcal{O}_K) \otimes_{\mathcal{O}_K} K$$

We finish this section with the following proposition

Proposition 3.14. *The following* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representations are admissible:*
a) $\widehat{H}^1_{X_{ord}}(X(N), K) = \varprojlim_s \varinjlim_m H^1_{X_{ord}}(X(Np^m)^{an}, \mathcal{O}_K/\varpi^s\mathcal{O}_K) \otimes_{\mathcal{O}_K} K$
b) $\widehat{H}^1_{X_{ord}}(X(N), \bar{\mathbb{F}}_p) = \varinjlim_m H^1_{X_{ord}}(X(Np^m)^{an}, \bar{\mathbb{F}}_p)$

Proof. This follows from the exact sequence (we use the notations from the previous section)

$$H^0(X(Np^m)_{ss}, A) \to H^1_{X_{ord}}(X(Np^m)^{an}, A) \to H^1(X(Np^m)^{an}, A)$$

and Proposition 3.9. □

Remark 3.15. *We have used above the fact that the cohomology groups* H^i_c *and* H^i *of the supersingular locus are smooth* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representations for all i. This is clear, because the cohomology groups of the Lubin-Tate tower are smooth. We also remark that one could prove that the mod p cohomology of the Drinfeld tower is smooth as is done in Lemma 6(i) from* [Har]*.*

4. Mod p and p-adic local Langlands correspondence

In this section, we sum up what is known about the mod p and p-adic local Langlands correspondence in a way which will be useful for us later.

4.1. **Mod p correspondence.** Let ω_n be the fundamental character of Serre of level n which is defined on inertia group I via $\sigma \mapsto \frac{\sigma(p^{1/n})}{p^{1/n}}$. Let ω be the mod p cyclotomic character. For $h \in \mathbb{N}$, we write Ind ω_n^h for the unique semisimple $\bar{\mathbb{F}}_p$-representation of $G_{\mathbb{Q}_p}$ which has determinant ω^h and whose restriction to I is isomorphic to $\omega_n^h \oplus \omega_n^{ph} \oplus ... \oplus \omega_n^{p^{n-1}h}$. If $\chi : G_{\mathbb{Q}_p} \to k^\times$ is a character, we will denote by $\rho(r, \chi)$ the representation $\mathrm{Ind}(\omega_2^{r+1}) \otimes \chi$ which is absolutely irreducible if $r \in \{0, ..., p-1\}$. In fact, any absolutely irreducible representation of $G_{\mathbb{Q}_p}$ of dimension 2 is isomorphic to some $\rho(r, \chi)$ for $r \in \{0, ..., p-1\}$.

On the GL_2-side, one considers representations $\mathrm{Sym}^r k^2$ inflated to $\mathrm{GL}_2(\mathbb{Z}_p)$ and then extended to $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times$ by making p acts by identity. We then consider the induced representation $\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{Sym}^r k^2$. One can show that the endomorphism ring (a Hecke algebra) $\mathrm{End}_{k[\mathrm{GL}_2(\mathbb{Q}_p)]}(\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{Sym}^r k^2)$ is isomorphic to $k[T]$, where T corresponds to the double class $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times \left(\begin{smallmatrix} p & 1 \\ 0 & 1 \end{smallmatrix}\right) \mathrm{GL}_2(\mathbb{Z}_p)$. For a character $\chi : G_{\mathbb{Q}_p} \to k^\times$ and $\lambda \in k$. we introduce representations:

$$\pi(r, \lambda, \chi) = \frac{\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{Sym}^r k^2}{T - \lambda} \otimes (\chi \circ \det)$$

For $r \in \{0, ..., p-1\}$ such that $(r, \lambda) \notin \{(0, \pm 1), (p-1, \pm 1)\}$, the representation $\pi(r, \lambda, \chi)$ is irreducible. If $\lambda = \pm 1$, then $\pi(r, \lambda, \chi)$ appears as either a subrepresentation or a subquotient special representation Sp. One proves that $\chi \circ \det$, $\mathrm{Sp} \otimes (\chi \circ \det)$ and $\pi(r, \lambda, \chi)$ for $r \in \{0, ..., p-1\}$ and $(r, \lambda) \notin \{(0, \pm 1), (p-1, \pm 1)\}$ are all the smooth irreducible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$.

This explicit description gives a mod p correspondence by associating $\rho(r, \chi)$ to $\pi(r, 0, \chi)$. We remark that the above construction is impossible to make for $\mathrm{GL}_2(F)$ when $F \neq \mathbb{Q}_p$, as there are much more representations from GL_2-side (see [BP]).

4.2. **Colmez's functor.** The p-adic correspondence is much more complicated and the two sides are not related directly. We will not recall the construction explicitly, but we mention the Colmez's functor which is a certain deformation functor, which when restricted to representations with coefficients in finite extensions of $\mathbb{Q}_p$ gives the p-adic local Langlands correspondence. For the general formalism behind Colmez functors, see [Cho].

The Colmez's functor (defined in [Co]) is an exact covariant functor $V : \mathrm{Rep}^f_{\mathrm{GL}_2(\mathbb{Q}_p)} \to \mathrm{Rep}_{G_{\mathbb{Q}_p}}$ from the category of smooth $\mathcal{O}$-representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ of finite length to the category of continuous $\mathcal{O}$-representations of $G_{\mathbb{Q}_p} = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. It is compatible with twists and reductions mod p. On the mod p level it gives the mod p local Langlands correspondence defined above.

Let π be a mod p supersingular representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ and let I_π be its injective envelope in $\mathrm{Rep}^f_{\mathrm{GL}_2(\mathbb{Q}_p)}$. Let $\rho = V(\pi)$, χ be the determinant of ρ and let ρ^{un} be the universal deformation of ρ with determinant χ. We denote by $\mathrm{Rep}^f_{\mathrm{GL}_2(\mathbb{Q}_p),(\pi)}$ the full subcategory of $\mathrm{Rep}^f_{\mathrm{GL}_2(\mathbb{Q}_p)}$ which consists of representations whose all irreducible subquotients are isomorphic to π. Paskunas has proved (see theorem 1.6 in [Pa1]) that for every object $\tau \in \mathrm{Rep}^f_{\mathrm{GL}_2(\mathbb{Q}_p),(\pi)}$

$$V(\tau)^\vee \simeq \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\tau, I_\pi) \otimes_{\tilde{E}} \rho^{un}$$

where $\tilde{E} = \mathrm{End}_{\mathrm{GL}_2(\mathbb{Q}_p)}(I_\pi)$ and where we have denoted by $(-)^\vee$ the Pontryagin dual, defined by $\mathrm{Hom}_{\mathcal{O}}(-, K/\mathcal{O})$. The Pontryagin duality furnishes an anti-equivalence between the category of smooth $\mathcal{O}$-torsion representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ and the category of pro-augmented modules, that is the category of profinite $\mathcal{O}[[H]]$-modules with an action of $\mathcal{O}[\mathrm{GL}_2(\mathbb{Q}_p)]$ such that the two actions are the same when restricted to $\mathcal{O}[H]$, where H is any compact open subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$.

We remark that the Colmez's functor yields an equivalence between the category $\mathrm{Rep}^f_{\mathrm{GL}_2(\mathbb{Q}_p),(\pi)}$ and the category of deformations of ρ with determinant equal to χ (see [Pa1]). Moreover this equivalence gives an isomorphism of $\tilde{E}$ with the deformation ring R^χ_ρ representating the deformation problem of ρ with determinant χ.

5. Supersingular representations

In this section we recall results on the structure of admissible representations and we apply them to the exact sequence of cohomology groups that we have introduced before, getting the first comparison between the cohomology of the Lubin-Tate tower and the cohomology of the tower of modular curves.

Let us a fix a supersingular representation π of $\mathrm{GL}_2(\mathbb{Q}_p)$ on a $\bar{\mathbb{F}}_p$-vector space with a central character ξ. Recall the following results of Paskunas:

Proposition 5.1. *Let τ be an irreducible smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ admitting a central character. If $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi, \tau) \neq 0$ then $\tau \simeq \pi$.*

Proof. See [Pa2] and [Pa3] for the case $p = 2$. □

This result permits us to consider blocks of representations and define a localisation with respect to a given supersingular representation. The general result of Gabriel on a block decomposition of locally finite categories gives:

Proposition 5.2. *We have a decomposition:*

$$\mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p),\xi}(\bar{\mathbb{F}}_p) = \mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p),\xi}(\bar{\mathbb{F}}_p)_{(\pi)} \oplus \mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p),\xi}(\bar{\mathbb{F}}_p)^{(\pi)}$$

where $\mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p),\xi}(\bar{\mathbb{F}}_p)$ *is the (abelian) category of admissible* $\bar{\mathbb{F}}_p$*-representations admitting a central character* ξ*,* $\mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p),\xi}(\bar{\mathbb{F}}_p)_{(\pi)}$ *(resp.* $\mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p),\xi}(\bar{\mathbb{F}}_p)^{(\pi)}$*) is the subcategory of it consisting of representations* Π *whose all the irreducible subquotients are (resp. are not) isomorphic to* π*.*

Proof. See Proposition 5.32 in [Pa1]. □

A similar result holds for K-Banach admissible representations. This permit us to consider the localisation functor with respect to π

$$V \mapsto V_{(\pi)}$$

on the category of admissible representations such that all irreducible subquotients of $V_{(\pi)}$ (or a reduction of a lattice in $V_{(\pi)}$ for K-Banach representations) are isomorphic to the fixed π. We can also consider the localisation of admissible representations over $\mathcal{O}_K/\varpi^s\mathcal{O}_K$.

Remark 5.3. *We note that the condition on the existence of central characters is not important. Central characters always exist by the work of Berger (*[Be1]*) in the mod p case, and by the work of Dospinescu-Schraen (*[DS]*) in the p-adic case.*

Later on, we will also show how one can localise smooth representations (which are not neccessarily admissible) with respect to π. This will be a localisation in the weak sense, meaning that we will obtain for a smooth $\bar{\mathbb{F}}_p$-representation V an exact sequence

$$0 \to V^{(\pi)} \to V \to V_{(\pi)} \to 0$$

where $V_{(\pi)}$ (respectively, $V^{(\pi)}$) is such that all (resp. none of) its irreducible subquotient representations are isomorphic to π. The difference is that the above exact sequence is not neccessarily split, at least we could not prove splitness by using our methods.

5.1. **Cohomology with compact support.** Let A be either $\mathcal{O}_K/\varpi^s\mathcal{O}_K$ or $\bar{\mathbb{F}}_p$ in the following. We apply the localisation functor to the exact sequence of admissible representations obtained from 2.1:

$$0 \to \widehat{\mathrm{Ker}}(N,A) \to \widehat{H}^1_c(X(N)_{ss},A) \to \widehat{H}^1(X(N),A) \to \widehat{\mathrm{Im}}(N,A) \to 0$$

getting

$$0 \to \widehat{\mathrm{Ker}}(N,A)_{(\pi)} \to \widehat{H}^1_c(X(N)_{ss},A)_{(\pi)} \to \widehat{H}^1(X(N),A)_{(\pi)} \to \widehat{\mathrm{Im}}(N,A)_{(\pi)} \to 0$$

Recall now, that after 2.2, $\widehat{\mathrm{Ker}}(N,A)$ is an admissible representation onto which surjects the induced representation

$$\mathrm{Ind}_{B_m(\infty)}^{\mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z})}\left(\bigoplus_a im(H^0(\mathbb{X}_{a,\infty},A) \to H^1_c(X(Np^m)_{ss},A))\right)$$

which is also admissible (as H^0 is finite). On these representations unipotent groups acts trivially by lemmas 3.7 and 3.8, and hence we see that both are induced from the tensor product of characters. This means that after localisation at π both these representations vanish

$$\mathrm{Ind}_{B_m(\infty)}^{\mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z})}\left(\bigoplus_a im(H^0(\mathbb{X}_{a,\infty},A) \to H^1_c(X(Np^m)_{ss},A))\right)_{(\pi)} = \widehat{\mathrm{Ker}}(N,A)_{(\pi)} = 0$$

and we arrive at

Theorem 5.4. *We have an injection of representations*

$$\widehat{H}^1_c(X(N)_{ss}, \mathcal{O}_K/\varpi^s\mathcal{O}_K)_{(\pi)} \hookrightarrow \widehat{H}^1(X(N), \mathcal{O}_K/\varpi^s\mathcal{O}_K)_{(\pi)}$$

and

$$\widehat{H}^1_c(X(N)_{ss}, \bar{\mathbb{F}}_p)_{(\pi)} \hookrightarrow \widehat{H}^1(X(N), \bar{\mathbb{F}}_p)_{(\pi)}$$

hence also, by taking limit

$$\widehat{H}^1_c(X(N)_{ss}, K)_{(\pi)} \hookrightarrow \widehat{H}^1(X(N), K)_{(\pi)}$$

By taking yet another direct limit, for $A = K, \bar{\mathbb{F}}_p$, we define

$$\widehat{H}^1_{ss,c,A} = \varinjlim_N \widehat{H}^1_c(X(N)_{ss}, A)$$

$$\widehat{H}^1_A = \varinjlim_N \widehat{H}^1(X(N), A)$$

Corollary 5.5. *We have an injection of representations*

$$(\widehat{H}^1_{ss,c,K})_{(\pi)} \hookrightarrow (\widehat{H}^1_K)_{(\pi)}$$

and

$$(\widehat{H}^1_{ss,c,\bar{\mathbb{F}}_p})_{(\pi)} \hookrightarrow (\widehat{H}^1_{\bar{\mathbb{F}}_p})_{(\pi)}$$

Remark 5.6. *This reasoning can give us also an injection for the cohomology groups at the integral level.*

We define also for a future use

$$\widehat{H}^1_{ord,K} = \varinjlim_N \varprojlim_s \varinjlim_m H^1(X(Np^m)_{ord}, \mathcal{O}_K/\varpi^s\mathcal{O}_K) \otimes_{\mathcal{O}_K} K$$

$$\widehat{H}^1_{ord,\bar{\mathbb{F}}_p} = \varinjlim_N \varinjlim_m H^1(X(Np^m)_{ord}, \bar{\mathbb{F}}_p)$$

and for $a \in \mathbb{Z}_p^\times$

$$\widehat{H}^1_{a,\infty,K} = \varinjlim_N \varprojlim_s \varinjlim_m H^1(\mathbb{X}_{a,\infty}(Np^m), \mathcal{O}_K/\varpi^s\mathcal{O}_K) \otimes_{\mathcal{O}_K} K$$

$$\widehat{H}^1_{a,\infty,\bar{\mathbb{F}}_p} = \varinjlim_N \varinjlim_m H^1(\mathbb{X}_{a,\infty}(Np^m), \bar{\mathbb{F}}_p)$$

5.2. **Cohomology without support.** We can apply similar reasoning as above to the situation without compact support. The roles of the ordinary locus and the supersingular locus are interchanged. By using again the decomposition of the ordinary locus, we get that the localisation of $\widehat{H}^1_{X_{ord}}$ vanishes

$$\widehat{H}^1_{X_{ord}}(X(N), A)_{(\pi)} = 0$$

and hence we get

Theorem 5.7. *We have an injection of representations*

$$(\widehat{H}^1_K)_{(\pi)} \hookrightarrow \widehat{H}^1_{ss,K}$$

and

$$(\widehat{H}^1_{\bar{\mathbb{F}}_p})_{(\pi)} \hookrightarrow \widehat{H}^1_{ss,\bar{\mathbb{F}}_p}$$

where $\widehat{H}^1_{ss,K}$ and $\widehat{H}^1_{ss,\bar{\mathbb{F}}_p}$ are defined similarly as above.

Later on, we will show that we have short exact sequences (we remark that considered representations are non-admissible, hence we have to first explain what does it mean to localise a non-admissible smooth representation)

$$0 \to (\widehat{H}^1_A)_{(\pi)} \to (\widehat{H}^1_{ss,A})_{(\pi)} \to (\widehat{H}^2_{X_{ord},A})_{(\pi)} \to 0$$

where $A = K$ or $\bar{\mathbb{F}}_p$ and

$$\widehat{H}^2_{X_{ord},A} = \varinjlim_N \widehat{H}^2_{X_{ord}}(X(N), A)$$

We will come back to these exact sequences after we prove some preliminary results on the structure of representations of quaternion algebras. Let us finish by giving another definition for a future use (where $a \in \mathbb{Z}_p^\times$)

$$\widehat{H}^1_{\mathbb{X}_{a,\infty},K} = \varinjlim_N \varprojlim_s \varinjlim_m H^1_{\mathbb{X}_{a,\infty}(Np^m)}(X(Np^m)^{an}, \mathcal{O}_K/\varpi^s\mathcal{O}_K) \otimes_{\mathcal{O}_K} K$$

$$\widehat{H}^1_{\mathbb{X}_{a,\infty},\bar{\mathbb{F}}_p} = \varinjlim_N \varinjlim_m H^1_{\mathbb{X}_{a,\infty}(Np^m)}(X(Np^m)^{an}, \bar{\mathbb{F}}_p)$$

6. New vectors

Because there does not exist at the moment the Colmez functor in the context of quaternion algebras, which would be similar to the one considered for example in [Pa1], we are forced to give a global definition of the mod p and p-adic Jacquet-Langlands correspondence. To do that, we prove an analogue of a classical theorem of Casselman in the context of the modified mod l Langlands correspondence of Emerton-Helm (see [EH]), which amounts to a statement that for any prime $l \neq p$, and for any local l-adic two-dimensional Galois representation ρ, there exists a compact, open subgroup $K_l \subset \mathrm{GL}_2(\mathbb{Z}_l)$ such that $\pi_l(\rho)^{K_l}$ has dimension 1, where $\pi_l(\rho)$ is the mod p representation of $\mathrm{GL}_2(\mathbb{Q}_l)$ associated to ρ by [EH].

Let $\mathfrak{b}$ be an ideal of $\mathbb{Z}_p$ and put $\Gamma_0(\mathfrak{b}) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \mathrm{GL}_2(\mathbb{Z}_p) | c \equiv 0 \mod \mathfrak{b}\}$. Let us recall the classical result of Casselman (see [Cas]):

Theorem 6.1. *Let π be an irreducible admissible infinite-dimensional representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\bar{\mathbb{Q}}_l$-vector space and let ϵ be the central character of π. Let $c(\pi)$ be the conductor of π which is the largest ideal of $\mathbb{Z}_p$ such that the space of vector v with $\pi(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})v = \epsilon(a)v$, for all $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma_0(c(\pi))$ is not empty. Then this space has dimension one.*

We will prove that the result holds also modulo l for the modified mod l Langlands correspondence:

Theorem 6.2. *Let $\pi = \pi(\rho)$ be the mod p admissible representation of $\mathrm{GL}_2(\mathbb{Q}_l)$ associated by the modified mod l Langlands correspondence to a Galois representation $\rho : G_{\mathbb{Q}_l} \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$. Then there exists an open, compact subgroup K of $\mathrm{GL}_2(\mathbb{Q}_l)$ such that $\dim_{\bar{\mathbb{F}}_p} \pi^K = 1$.*

Proof. We recall the results of [EH] concerning the construction of the modified mod l Langlands correspondence. By Proposition 5.2.1 of [EH], the theorem is true when ρ^{ss} is not a twist of $1 \oplus |\cdot|$, by the reduction modulo p of the classical result of Casselman from [Cas], which in the $l \neq p$ situation was proved by Vigneras in [Vi2] (see Theorem 23 and Proposition 24). When this is not the case, we can suppose that in fact $\rho^{ss} = 1 \oplus |\cdot|$ and we go by case-by-case analysis of the possible forms of $\pi(\rho)$ as described in [EH] after Proposition 5.2.1 and in [He]. The $\pi(\rho)$'s which appear are mostly extensions of four kinds of representations (and some combinations of them): trivial representation 1, $|\cdot| \circ \det$, the Steinberg St, $\pi(1)$ of Vigneras (see [Vi2]).

1) Suppose $0 \to \pi(1) \to \pi(\rho) \to 1 \to 0$. In this case $l \equiv -1 \mod p$. Let $\Gamma_0(p) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \mathrm{GL}_2(\mathbb{Z}_p) | c \equiv 0 \mod p, a \equiv d \equiv 1 \mod p\}$. Then we have a long exact sequence associated with higher invariants by $\Gamma_0(p)$:

$$0 \to \pi(\rho)^{\Gamma_0(p)} \to 1 \to R^1\pi(1)^{\Gamma_0(p)}$$

as $\pi(1)^{\Gamma_0(p)} = 0$ by the Proposition 24 of [Vi2]. We conclude by observing that $R^1\pi(1)^{\Gamma_0(p)} = 0$ because $|\Gamma_0(p)| = p^\infty \cdot (p-1)$ and $l \not| |\Gamma_0(p)|$ by our assumption.

2) In the same way we deal with the situation when $\pi(\rho)$ is an extension of $|\cdot|\circ\det$ by $\pi(1)$ with the same assumption on l.

3) When $l\equiv -1 \mod p$ it is also possible to have $0\to\pi(1)\to\pi(\rho)\to 1\oplus|\cdot|\circ\det\to 0$. Look at $\mathrm{GL}_2(\mathbb{Z}_p)$-invariants. The associated long exact sequence is

$$0\to\pi(\rho)^{\mathrm{GL}_2(\mathbb{Z}_p)}\to(1\oplus|\cdot|\circ\det)^{\mathrm{GL}_2(\mathbb{Z}_p)}\to R^1\pi(1)^{\mathrm{GL}_2(\mathbb{Z}_p)}=\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{F}_p)}(1,\pi(1))$$

Because $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{F}_p)}(1,\pi(1))=\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{F}_p)}(|\cdot|\circ\det,\pi(1))$, the map $(1\oplus|\cdot|\circ\det)^{\mathrm{GL}_2(\mathbb{Z}_p)}\to\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{F}_p)}(1,\pi(1))$ gives a line in $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{F}_p)}(1,\pi(1))$ and hence the kernel, i.e. $\pi(\rho)^{\mathrm{GL}_2(\mathbb{Z}_p)}$, is also one-dimensional as $(1\oplus|\cdot|\circ\det)^{\mathrm{GL}_2(\mathbb{Z}_p)}$ has dimension two.

4) The last case with which we have to deal is the case when p is odd, $l\equiv 1 \mod p$ and we have an extension:

$$0\to\mathrm{St}\to\pi(\rho)\to 1\to 0$$

In this case $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(1,\mathrm{St})$ is two-dimensional (see Lemma 4.2 in [He]) and let W, W' be two non-isomorphic extensions of 1 by St (one of them can be taken to be $\pi(\rho)$). Let us take V to be the "‘universal"’ extension of 1 by St (it is the unique extension of $1\oplus 1$ by St, which can be also constructed as the pushout of W and W' over St) - see section 4 of [He]. By Proposition 4.3 in [He], V is an essentially AIG envelope of St which implies that the socle of V is isomorphic to St (for the precise definition, see [EH]). We see that there are exactly two copies of St embedded into V. Let K be the Iwahori subgroup so St^K is one-dimensional. We claim that V^K is two-dimensional. Indeed, we have

$$0\to\mathrm{St}^K\to V^K\to 1^K\oplus 1^K\to R^1\,\mathrm{St}^K=\mathrm{Ext}^1(1,\mathrm{St}^K)$$

The map $1^K\oplus 1^K\to\mathrm{Ext}^1(1,\mathrm{St}^K)$ gives a line in $\mathrm{Ext}^1(1,\mathrm{St}^K)$ and hence we see that V^K is of dimension two. This means that both W^K and $(W')^K$ which are embedded in V^K are of dimension 1 as they are essentially AIG and also have their socles isomorphic to St. ☐

7. The fundamental representation

Following the original Deligne's approach to the non-abelian Lubin-Tate theory, we define the local fundamental representation. Using it, we refine the Lubin-Tate side of the injections we have considered. Then we recall Emerton's results on the cohomology of the tower of modular curves, yielding by a comparison an information on the local fundamental representation. Our arguments are similar to those given in [De].

7.1. Cohomology of the supersingular tube.

We have introduced in section 2.3, the set Δ, spaces $LT_{\Delta/K_m}=\coprod_{\delta\in\Delta/K_m}LT_\delta$ and we have obtained a description of the supersingular tube

$$X(Np^m)_{ss}\simeq LT_{\Delta/K_m}\times_{D^\times(\mathbb{Q})}\mathrm{GL}_2(\mathbb{A}_f^p)/K(N)$$

Definition 7.1. *Define the fundamental representation* H^1_{LT} *by*

$$\widehat{H}^1_{LT,c,\bar{\mathbb{F}}_p}=\varinjlim_m H^1_c(LT_{\Delta/K_m},\bar{\mathbb{F}}_p)$$

and

$$\widehat{H}^1_{LT,c,K}=\varprojlim_s\varinjlim_m H^1_c(LT_{\Delta/K_m},\mathcal{O}_K/\varpi^s\mathcal{O}_K)\otimes_{\mathcal{O}_K}K$$

Similarly we introduce the fundamental representation without support denoting it by $\widehat{H}^1_{LT,\bar{\mathbb{F}}_p}$ *and* $\widehat{H}^1_{LT,K}$ *respectively.*

From the description of supersingular points, we have for $A=\mathcal{O}_K/\varpi^s\mathcal{O}_K,\bar{\mathbb{F}}_p$

$$H^1_c(X(Np^m)_{ss},A)=H^1_c(LT_{\Delta/K_m}\times_{D^\times(\mathbb{Q})}\mathrm{GL}_2(\mathbb{A}_f^p)/K(N),A)=$$

$$=\{f:D^\times(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_f^p)/K(N)\to H^1_c(LT_{\Delta/K_m},A)\}$$

We take a direct limit:

$$\varinjlim_m H^1_c(X(Np^m)_{ss}, A) \simeq \{f : D^\times(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}^p_f)/K(N) \to \varinjlim_m H^1_c(LT_{\Delta/K_m}, A)\}$$

Take $A = \bar{\mathbb{F}}_p$ and then also a limit over N to obtain

$$\widehat{H}^1_{ss,c,\bar{\mathbb{F}}_p} \simeq \{f : D^\times(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}^p_f) \to \widehat{H}^1_{LT,\bar{\mathbb{F}}_p}\} \simeq$$

$$\simeq \{f : D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f) \to \widehat{H}^1_{LT,c,\bar{\mathbb{F}}_p}\}^{D^\times(\mathbb{Q}_p)} \simeq$$

$$\simeq \left(\{f : D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f) \to \bar{\mathbb{F}}_p\} \otimes_{\bar{\mathbb{F}}_p} \widehat{H}^1_{LT,c,\bar{\mathbb{F}}_p}\right)^{D^\times(\mathbb{Q}_p)}$$

Let

$$\mathbf{F} = \{f : D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f) \to \bar{\mathbb{F}}_p\}$$

where f are locally constant functions, then

$$\widehat{H}^1_{ss,c,\bar{\mathbb{F}}_p} \simeq \left(\mathbf{F} \otimes_{\bar{\mathbb{F}}_p} \widehat{H}^1_{LT,c,\bar{\mathbb{F}}_p}\right)^{D^\times(\mathbb{Q}_p)} \tag{1}$$

By a similar reasoning we also get

$$\widehat{H}^1_{ss,c,K} \simeq \left(\widehat{\mathbf{F}} \otimes_K \widehat{H}^1_{LT,c,K}\right)^{D^\times(\mathbb{Q}_p)}$$

where

$$\widehat{\mathbf{F}} = \varinjlim_{K^p} \varprojlim_s \varinjlim_{K_p} H^0(D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)/K_pK^p, \mathcal{O}_K/\varpi^s\mathcal{O}_K) \otimes_{\mathcal{O}_K} K$$

we neglect dependence on K in the notation as it will be clear from the context. Of course, we get similar results for the cohomology without support

$$\widehat{H}^1_{ss,\bar{\mathbb{F}}_p} \simeq \left(\mathbf{F} \otimes_{\bar{\mathbb{F}}_p} \widehat{H}^1_{LT,\bar{\mathbb{F}}_p}\right)^{D^\times(\mathbb{Q}_p)}$$

$$\widehat{H}^1_{ss,K} \simeq \left(\widehat{\mathbf{F}} \otimes_K \widehat{H}^1_{LT,K}\right)^{D^\times(\mathbb{Q}_p)}$$

7.2. **Emerton's results.** We recall Emerton's results on the completed cohomology of modular curves. Remark that we are using implicitly the comparison theorem for étale cohomology of a scheme and its analytification which is proved in [Ber1].

Let us fix a finite set $\Sigma = \Sigma_0 \cup \{p\}$. Let $K^\Sigma = \prod_{l\notin\Sigma} K_l$ where $K_l = \mathrm{GL}_2(\mathbb{Z}_l)$ and choose an open, compact subgroup K_{Σ_0} of $\prod_{l\in\Sigma_0} \mathrm{GL}_2(\mathbb{Z}_l)$. Let $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ be an odd, irreducible, continuous representation unramified outside Σ. Remark that by Serre's conjecture (see [Kh1]) $\bar{\rho}$ is modular. Let us denote by $\mathfrak{m}$ the maximal ideal in $\mathbb{T}(K_{\Sigma_0})$ which corresponds to $\bar{\rho}$. We write also $\bar{\rho}_{|G_{\mathbb{Q}_p}} = \mathrm{Ind}\,\alpha$, where α can be considered as a character of $\mathbb{Q}^\times_{p^2}$ by the local class field theory. Let ρ be a global promodular lift of $\bar{\rho}$, i.e. ρ is a representation of $G_{\mathbb{Q}}$ on K, which corresponds to some closed point $\mathfrak{p}$ in the spectrum of the completed Hecke algebra $\mathrm{Spec}\,\mathbb{T}_{\mathfrak{m},\Sigma}$. For the definitions, see section 5 of [Em2].

Theorem 7.2. *Assuming that $\bar{\rho}$ satisfies certain technical hypotheses (see the proof below), we have an isomorphism*

$$\widehat{H}^1_{\bar{\mathbb{F}}_p}[\mathfrak{m}]^{K^\Sigma} \simeq \pi \otimes_{\bar{\mathbb{F}}_p} \pi_{\Sigma_0}(\bar{\rho}) \otimes_{\bar{\mathbb{F}}_p} \bar{\rho}$$

where π is a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to $\bar{\rho}$ by the mod p local Langlands correspondence and $\pi_{\Sigma_0}(\bar{\rho})$ is a representation $\mathrm{GL}_2(\mathbb{A}^{\Sigma_0}_f)$ associated to $\bar{\rho}$ by the modified local Langlands correspondence mod l for $l \in \Sigma_0$ (see [EH]*).*

Proof. For the exact assumptions, see Proposition 6.1.20 in [Em2]. Those assumptions are not important for our applications. □

Theorem 7.3. *Assuming that ρ satisfies certain technical hypotheses (see the proof below), we have an isomorphism*

$$\widehat{H}^1_K[\mathfrak{p}]^{K^\Sigma} \simeq B(\rho_{|G_{\mathbb{Q}_p}}) \otimes_K \pi_{\Sigma_0}(\rho) \otimes_K \rho$$

where $B(\rho_{|G_{\mathbb{Q}_p}})$ is a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to ρ by the p-adic local Langlands correspondence, and $\pi_{\Sigma_0}(\rho)$ is a representation $\mathrm{GL}_2(\mathbb{A}_f^{\Sigma_0})$ associated to ρ by the classical l-adic local Langlands correspondence for $l \in \Sigma_0$.

Proof. For the exact assumptions, see the proof of Proposition 6.1.17 in [Em2]. Those assumptions are not important for our applications. □

7.3. **Comparison.** We will use results of Emerton to describe a part of $\widehat{H}^1_{ss,\bar{\mathbb{F}}_p}$. We start by comparing mod p Hecke algebras for GL_2 and for $D^\times$. On both $\mathbf{F}$ and $\widehat{\mathbf{F}}$, after taking K^Σ-invariants, there is an action of a Hecke algebra. For $l \notin \Sigma$, we have a Hecke character T_l acting on functions of $D^\times(\mathbb{A}_f)$ by

$$T_l(f)(x) = f(xg) + \sum_{i=0}^{l-1} f(xg_i)$$

where $g = \left(\begin{smallmatrix} l & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $g_i = \left(\begin{smallmatrix} 1 & 0 \\ i & l \end{smallmatrix}\right)$ are both considered as elements of $D^\times(\mathbb{A}_f)$ having 1 at places different from l. Let us denote by $\mathbb{T}^D(K_{\Sigma_0}) = \varprojlim_{K_p} \mathbb{T}^D(K_pK_{\Sigma_0}K^\Sigma)$ the (completed) Hecke algebra, which is a free $\mathcal{O}$-algebra spanned by the operators T_l and S_l for all $l \notin \Sigma$, where $S_l = [K_{\Sigma_0}K^\Sigma\left(\begin{smallmatrix} \varpi & 0 \\ 0 & \varpi \end{smallmatrix}\right)K_{\Sigma_0}K^\Sigma]$. By the results of Serre (see letter to Tate from [Se]), systems of eigenvalues for (T_l) of $\mathbb{T}^D(K_{\Sigma_0})$ on $\mathbf{F}$ are in bijection with systems of eigenvalues for (T_l) of $\mathbb{T}(K_{\Sigma_0})$ coming from mod p modular forms. This allows us to identify maximal ideals of $\mathbb{T}^D(K_{\Sigma_0})$ with those of $\mathbb{T}(K_{\Sigma_0})$ and in what follows we will make no distinction between them.

Let $\bar{\rho}_p$ be the local Galois representation associated to a supersingular representation π of $\mathrm{GL}_2(\mathbb{Q}_p)$ by the mod p Langlands correspondence. We assume that there exists a representation $\bar{\rho} : G_\mathbb{Q} \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ which is odd, irreducible, continuous, unramified outside a finite set $\Sigma = \Sigma_0 \cup \{p\}$, and such that $\bar{\rho}_{|D_p} = \bar{\rho}_p$. Let us denote by $\mathfrak{m}$ the maximal ideal in the Hecke algebra $\mathbb{T}(K_{\Sigma_0})$ corresponding to $\bar{\rho}$. Moreover we assume that results of Emerton apply to $\bar{\rho}$. We denote by $K_\mathfrak{m}^{\Sigma_0}$ an open compact subgroup of $\prod_{l\in\Sigma_0} \mathrm{GL}_2(\mathbb{Z}_l)$ for which $\pi_{\Sigma_0}(\bar{\rho})^{K_\mathfrak{m}^{\Sigma_0}}$ is a one-dimensional vector space (new vectors). We put $K_\mathfrak{m} = K_\mathfrak{m}^{\Sigma_0}K^\Sigma$ and we define:

$$\sigma_\mathfrak{m} = \mathbf{F}[\mathfrak{m}]^{K_\mathfrak{m}}$$

This is a representation of $D^\times(\mathbb{Q}_p)$. Taking $K_\mathfrak{m}$-invariants which commute with $D^\times(\mathbb{Q}_p)$-invariants, we get

$$\left(\widehat{H}^1_{ss,\bar{\mathbb{F}}_p}\right)^{K_\mathfrak{m}} \simeq \left(\mathbf{F}^{K_\mathfrak{m}} \otimes_{\bar{\mathbb{F}}_p} \widehat{H}^1_{LT,\bar{\mathbb{F}}_p}\right)^{D^\times(\mathbb{Q}_p)}$$

Taking $[\mathfrak{m}]$-part and localising at (π) we get:

$$\left(\widehat{H}^1_{ss,\bar{\mathbb{F}}_p}[\mathfrak{m}]\right)^{K_\mathfrak{m}}_{(\pi)} \simeq \widehat{H}^1_{LT,\bar{\mathbb{F}}_p,(\pi)}[\sigma_\mathfrak{m}^\vee]$$

Thus, by the results proven earlier, we have

$$\pi \otimes_{\bar{\mathbb{F}}_p} \bar{\rho} \simeq \left(\widehat{H}^1_{\bar{\mathbb{F}}_p}[\mathfrak{m}]\right)^{K_\mathfrak{m}}_{(\pi)} \hookrightarrow \left(\widehat{H}^1_{ss,\bar{\mathbb{F}}_p}[\mathfrak{m}]\right)^{K_\mathfrak{m}}_{(\pi)} \simeq \widehat{H}^1_{LT,\bar{\mathbb{F}}_p,(\pi)}[\sigma_\mathfrak{m}^\vee]$$

and we arrive at

Theorem 7.4. *We have an $\mathrm{GL}_2(\mathbb{Q}_p) \times G_{\mathbb{Q}_p}$-equivariant injection:*

$$\pi \otimes_{\bar{\mathbb{F}}_p} \bar{\rho} \hookrightarrow \widehat{H}^1_{LT,\bar{\mathbb{F}}_p,(\pi)}[\sigma_\mathfrak{m}^\vee]$$

We will strengthen this result after proving additional facts about $\sigma_\mathfrak{m}$. Later on, we will obtain the analogous result in the p-adic setting. We will also say, what happens when one would take in the above cohomology groups with compact support.

7.4. **The mod p Jacquet-Langlands correspondence.** We have defined above

$$\sigma_{\mathfrak{m}} = \mathbf{F}[\mathfrak{m}]^{K_{\mathfrak{m}}}$$

This is a mod p representation of $D^\times(\mathbb{Q}_p)$ which is one of our candidates for the mod p Jacquet-Langlands correspondence we search for. We will analyse this representation more carefully in the next section, getting a result about its socle. The question we do not answer here is whether this local representation is independent of the Hecke ideal $\mathfrak{m}$ and if yes, how to construct it by local means. We make a natural conjecture

Conjecture 7.5. *Let $\mathfrak{m}$ and $\mathfrak{m}'$ be two maximal ideals of the Hecke algebra, which correspond to Galois representations $\bar{\rho}$ and $\bar{\rho}'$ such that $\bar{\rho}_p \simeq \bar{\rho}'_p$. Then we have a $D^\times(\mathbb{Q}_p)$-equivariant isomorphism*

$$\sigma_{\mathfrak{m}} \simeq \sigma_{\mathfrak{m}'}$$

This conjecture is natural in the view of the fact, that $\sigma_{\mathfrak{m}}$ should play a role of the mod p Jacquet-Langlands correspondence and it should depend only on a local data. In fact, this conjecture is implied by another conjecture - the local-global compatibility part of the Buzzard-Diamond-Jarvis conjecture (see Conjecture 4.7 in [BDJ])

Conjecture 7.6. *We have a $D^\times(\mathbb{A})$-equivariant isomorphism*

$$\mathbf{F}[\mathfrak{m}] \simeq \sigma \otimes \pi^p(\bar{\rho})$$

where σ is a $D^\times(\mathbb{Q}_p)$-representation which depends only on $\bar{\rho}_p$, where $\bar{\rho}$ is the Galois representation associated to $\mathfrak{m}$.

The conjecture of Buzzard-Diamond-Jarvis would be proved if one could show an existence of an analogue of the Colmez functor in the context of quaternion algebras (see [Cho] for a general definition of a Colmez functor). Then, the methods of Emerton from [Em2] could be applied to give a proof.

8. Representations of quaternion algebras: mod p theory

In this section we analyse more carefully mod p representations of quaternion algebras, especially representations $\sigma_{\mathfrak{m}}$ defined in the preceding section. We also define a naive mod p Jacquet-Langlands correspondence.

8.1. **Naive mod p Jacquet-Langlands correspondence.** By the work of Vigneras (see [Vi1]), we know that all irreducible representation of $D^\times$ are of dimension 1 or 2 and are either
1) a character of $D^\times(\mathbb{Q}_p)$, or
2) are of the form $\mathrm{Ind}_{\mathcal{O}_D^\times \mathbb{Q}_{p^2}^\times}^{D^\times} \alpha$ where α is a character of $\mathbb{Q}_{p^2}^\times$.

Let $\bar{\rho}_p$ be the mod p 2-dimensional irreducible Galois representation which corresponds to the supersingular representation π of $\mathrm{GL}_2(\mathbb{Q}_p)$ by the mod p Local Langlands correspondence. It is of the form $\mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}}(\omega_2^r) \otimes \chi$ where χ is a character and $r \in \{1, ..., p\}$ (see [Br1]).

Definition 8.1. *The naive mod p Jacquet-Langlands correspondence is*

$$\mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}}(\omega_2^r) \otimes \chi \mapsto \mathrm{Ind}_{\mathcal{O}_D^\times \mathbb{Q}_{p^2}}^{D^\times}(\omega_2^r) \otimes \chi$$

where ω_2^r is treated as a character of $\mathbb{Q}_{p^2}$ by the local class field theory and χ is considered both as a character of $G_{\mathbb{Q}_p}$ and $D^\times(\mathbb{Q}_p)$. This gives a bijection between two-dimensional representations of $G_{\mathbb{Q}_p}$ and two-dimensional representations of $D^\times(\mathbb{Q}_p)$. Similar correspondence holds for characters.

For a character $\alpha : \mathbb{Q}_{p^2} \to \bar{\mathbb{F}}_p^\times$, we will denote by $\mathrm{Ind}\,\alpha$ either the representation of $G_{\mathbb{Q}_p}$ obtained by the local class field theory and an induction or a $D^\times(\mathbb{Q}_p)$-representation obtained via the naive mod p Jacquet-Langlands correspondence. It will be clear from the context which representation we mean.

8.2. **Quaternionic forms.** Let D be a quaternion algebra over $\mathbb{Q}$, ramified at p and at ∞. Let K be a finite extension of $\mathbb{Q}_p$ with the ring of integers $\mathcal{O}$ and a uniformiser ϖ. Define

$$\mathbf{F} = \varinjlim_{K} H^0(D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)/K, \bar{\mathbb{F}}_p)$$

$$\mathbf{F}_{\mathcal{O}} = \varinjlim_{K} H^0(D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)/K, \mathcal{O})$$

$$\widehat{\mathbf{F}}_{\mathcal{O}} = \varinjlim_{K^p} \varprojlim_{s} \varinjlim_{K_p} H^0(D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)/K_pK^p, \mathcal{O}/\varpi^s\mathcal{O})$$

Define also $\mathbf{F}_K = \mathbf{F}_{\mathcal{O}} \otimes_{\mathcal{O}} K$. We can make similar definitions for other $\mathbb{F}_p$-algebras (for example for finite extensions of $\mathbb{F}_p$ or for $\bar{\mathbb{Z}}_p$ in $\mathbf{F}_{\bar{\mathbb{Z}}_p}$ which we will use in the text).

Recall that we have fixed a finite set $\Sigma = \Sigma_0 \cup \{p\}$ and chosen an open, compact subgroup K_{Σ_0} of $\prod_{l\in\Sigma_0} \mathrm{GL}_2(\mathbb{Z}_l)$. On each of the above spaces, after taking K^Σ-invariants, there is an action of the Hecke algebra $\mathbb{T}^D(K_{\Sigma_0}) = \varprojlim_{K_p} \mathbb{T}^D(K_pK_{\Sigma_0}K^\Sigma)$. Recall also that we have defined $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ an odd, irreducible, continuous representation unramified outside Σ and we denoted by $\mathfrak{m}$ the maximal ideal in $\mathbb{T}(K_{\Sigma_0})$ (or in $\mathbb{T}^D(K_{\Sigma_0})$) which corresponds to $\bar{\rho}$. We write

$$\bar{\rho}_{|G_{\mathbb{Q}_p}} = \mathrm{Ind}\,\alpha$$

where α can be considered as a character of $\mathbb{Q}_{p^2}^\times$ by the local class field theory.

Proposition 8.2. *Take an open, compact subgroup K_p of $D^\times(\mathbb{Q}_p)$ and choose K_{Σ_0} to be an open, compact subgroup of $\prod_{l\in\Sigma_0} \mathrm{GL}_2(\mathbb{Z}_l)$ such that $K_pK_{\Sigma_0}K^\Sigma$ is neat. Then $\mathbf{F}_{\mathfrak{m}}^{K_{\Sigma_0}K^\Sigma}$ is injective as a smooth representation of K_p.*

We do not define the notion of neatness for which we refer to section 0.6 in [Pi]. We only need this condition to ensure that K_p acts freely as in the proof below. Any sufficiently small open compact subgroup is neat.

Proof. Let M be any smooth finitely generated representation of K_p. We have $\mathbf{F}^{K_{\Sigma_0}K^\Sigma} = \varinjlim_{K'_p} \mathbf{F}^{K'_pK_{\Sigma_0}K^\Sigma}$ where $K'_p \subset K_p$ runs over sufficiently small, normal open subgroups of $\mathcal{O}_D^\times$, so that K'_p acts trivially on M. We can associate to M a local system $\mathcal{M}$ on $D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)/K_{\Sigma_0}K^\Sigma$. Because K_p acts freely on $D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)/K_{\Sigma_0}K^\Sigma$ by the assumption of neatness, we can descend this system to each $D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)/K'_pK_{\Sigma_0}K^\Sigma$, where K'_p is as above. Moreover on each $D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)/K'_pK_{\Sigma_0}K^\Sigma$, $\mathcal{M}$ is a constant local system and hence:

$$\mathrm{Hom}_{K_p}(M, \mathbf{F}^{K_{\Sigma_0}K^\Sigma}) \simeq \varinjlim_{K'_p} \mathrm{Hom}_{K_p}(M, \mathbf{F}^{K'_pK_{\Sigma_0}K^\Sigma}) \simeq \varinjlim_{K'_p} (\mathbf{F}^{K'_pK_{\Sigma_0}K^\Sigma}(\mathcal{M}^\vee))^{K_p} \simeq \mathbf{F}^{K_p^\times K_{\Sigma_0}K^\Sigma}(\mathcal{M}^\vee)$$

where $\mathbf{F}(\mathcal{M}^\vee) = H^0(D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f), \mathcal{M}^\vee)$. Because $\mathbf{F}^{K_pK_{\Sigma_0}K^\Sigma}(\mathcal{M}^\vee)$ is an exact functor (there is no H^1), we get the result. □

The following lemma is ubiquitous in what follows and we will refer to it implicitely.

Lemma 8.3. *We have a Hecke-equivariant isomorphism*

$$\mathbf{F}_{\bar{\mathbb{Z}}_p} \otimes_{\bar{\mathbb{Z}}_p} \bar{\mathbb{F}}_p \simeq \mathbf{F}$$

Proof. It follows from the definition. For details see Lemma 7.4.1 in [EGH]. □

We will now start to analyse socles of quaternionic forms. Let us start with the following lemma:

Lemma 8.4. *Let β be an $\bar{\mathbb{F}}_p$-representation of $\mathcal{O}_D^\times$. We have $\mathrm{Hom}_{\mathcal{O}_D^\times}(\beta^\vee, \mathbf{F}_{\mathfrak{m}}^{K^p}) \simeq \mathbf{F}_{\mathfrak{m}}^{K^p}\{\beta\}$, where $\mathbf{F}_{\mathfrak{m}}^{K^p}\{\beta\}$ is the space of automorphic functions $D(\mathbb{Q})\backslash D(\mathbb{A}_f)/K^p \to \beta$.*

Proof. The isomorphism is given by an explicit map. See Lemma 7.4.3 in [EGH]. □

Proposition 8.5. *The only irreducible representations which appear in $\mathbf{F}_{\mathfrak{m}}^{K_{\Sigma_0}K^\Sigma}$ are isomorphic to $\sigma^\vee = (\mathrm{Ind}\,\alpha)^\vee$.*

Proof. Observe that the only representations which can appear in the $\mathcal{O}_D^\times$-socle of $\mathbf{F}_{\mathfrak{m}}^{K^p}$ are Serre weights of $\bar{\rho}^\vee$. This follows from the lemma above and the definition of being modular, i.e. $\bar{\rho}^\vee$ is modular of weight β (where β is a representation of $\mathcal{O}_D^\times$) if and only if there exists an open compact subset U of $D^\times(\mathbb{A}_f)$ such that $\mathbf{F}_{\mathfrak{m}}^U\{\beta\} \neq 0$. By the lemma, this is equivalent to $\mathrm{Hom}_{\mathcal{O}_D^\times}(\beta^\vee, \mathbf{F}_{\mathfrak{m}}^U)$ which holds if and only if $\beta \in soc_{\mathcal{O}_D^\times}\mathbf{F}_{\mathfrak{m}}^U$. Now the result follows from the result of Khare in [Kh2], as the only possible weights which can appear are $\alpha^\vee$ and $(\alpha^p)^\vee$. Hence the $D^\times(\mathbb{Q}_p)$-socle contains only $(\mathrm{Ind}\,\alpha)^\vee$. ∎

As a corollary we also get the $[\mathfrak{m}]$-isotypic analogue of the above

Corollary 8.6. *The only irreducible representations which appear in $\mathbf{F}^{K_{\Sigma_0}K^\Sigma}[\mathfrak{m}]$ are isomorphic to $\sigma^\vee = (\mathrm{Ind}\,\alpha)^\vee$.*

We are now ready to strengthen the theorem which appeared before

Theorem 8.7. *We have a $\mathrm{GL}_2(\mathbb{Q}_p) \times G_{\mathbb{Q}_p}$-equivariant injection*

$$\sigma \otimes \pi \otimes \bar{\rho} \hookrightarrow \widehat{H}^1_{LT,\bar{\mathbb{F}}_p,(\pi)}$$

Proof. We have proved earlier that we have

$$\pi \otimes \bar{\rho} \hookrightarrow \widehat{H}^1_{LT,\bar{\mathbb{F}}_p,(\pi)}[\sigma_{\mathfrak{m}}^\vee]$$

Hence also

$$\sigma_{\mathfrak{m}}^\vee \otimes \pi \otimes \bar{\rho} \hookrightarrow \sigma_{\mathfrak{m}}^\vee \otimes \widehat{H}^1_{LT,\bar{\mathbb{F}}_p,(\pi)}[\sigma_{\mathfrak{m}}^\vee]$$

Using the evaluation map (which is non-zero) and combining it with the fact that the only irreducible $D^\times(\mathbb{Q}_p)$-representation which appears in $\sigma_{\mathfrak{m}}^\vee$ is σ, we conclude. ∎

Let

$$n = \dim_{\bar{\mathbb{F}}_p} \mathrm{Hom}_{D^\times(\mathbb{Q}_p)}((\mathrm{Ind}\,\alpha)^\vee, \mathbf{F}_{\mathfrak{m}}^{K_{\Sigma_0}K^\Sigma})$$

It is conjectured that $n = 1$ (even in the more general setting, see Section 8 of [Br3]). Thus we would get that $\mathbf{F}_{\mathfrak{m}}^{K_{\Sigma_0}K^\Sigma}$ is isomorphic to an injective envelope of $\mathrm{Ind}\,\alpha$ as a representation of $D^\times(\mathbb{Q}_p)/p^{\mathbb{Z}}$. This would give us a particularly simple form of $\mathbf{F}_{\mathfrak{m}}^{K_{\Sigma_0}K^\Sigma}$ as we can compute the injective envelope explicitely in this case.

The injective envelope of a character α of $\mathcal{O}_D^\times$ is equal to $\mathrm{Ind}_{\mathbb{F}_{p^2}^\times}^{\mathcal{O}_D^\times}\alpha$, because we have a decomposition $\mathcal{O}_D^\times \simeq (1 + \pi\mathcal{O}_D^\times) \ltimes \mathbb{F}_{p^2}^\times$, where π is a uniformiser. As $1 + \pi\mathcal{O}_D^\times$ is pro-p, the injective envelope of α on it is $C^\infty(1 + \mathcal{O}_D^\times)$, and as order of $\mathbb{F}_{p^2}^\times$ is prime to p, any character is both injective and projective, hence the result. To obtain an injective envelope for a character of $D^\times(\mathbb{Q}_p)$, we have to include also an action of the center of $D^\times(\mathbb{Q}_p)$. The injective envelope of α is isomorphic to $\mathrm{Ind}_{\mathbb{F}_{p^2}^\times \times p^{\mathbb{Z}}}^{D^\times(\mathbb{Q}_p)}\alpha$ where on $p^{\mathbb{Z}}$ we have the trivial action (as we consider our representations without fixing a central character; if we were to work in the category with a fixed central character ξ, we would have an action of ξ on $p^{\mathbb{Z}}$).

Before moving further, let us mention another structure theorem for our $D^\times(\mathbb{Q}_p)$-representations, which shows that our mod p Jacquet-Langlands correspondence defined above is of entirely different nature that the one with complex coefficients.

Proposition 8.8. *The $D^\times(\mathbb{Q}_p)$-representation $\mathbf{F}^{K_{\Sigma_0}K^\Sigma}[\mathfrak{m}]$ is of infinite length.*

Proof. We give a sketch of a proof, which is contained in [BD] as Corollary 3.2.5 (it is conditional on the local-global compatibility part of the Buzzard-Diamond-Jarvis conjecture). Firstly observe that it is enough to prove that $\mathbf{F}^{K_{\Sigma_0}K^\Sigma}[\mathfrak{m}]$ is of infinite dimension over $\bar{\mathbb{F}}_p$, because a representation of finite length will be also of finite dimension as $D^\times$ is compact modulo center. Suppose now that we have an automorphic form π such that the reduction of its associated Galois representation $\bar{\rho}_\pi$ is isomorphic to $\bar{\rho}$ and $\pi^{K_{\Sigma_0}K^{\Sigma}} \neq 0$. Then there is a lattice $\Lambda_\pi = \mathbf{F}_{\bar{\mathbb{Z}}_p}^{K_{\Sigma_0}K^{\Sigma}} \cap \pi^{K_{\Sigma_0}K^{\Sigma}}$ inside $\pi^{K_{\Sigma_0}K^{\Sigma}}$. Its reduction $\bar{\Lambda}_\pi = \Lambda_\pi \otimes_{\bar{\mathbb{Z}}_p} \bar{\mathbb{F}}_p$ lies in $\mathbf{F}^{K_{\Sigma_0}K^{\Sigma}}[\mathfrak{m}]$ so it is enough to prove that we can find automorphic representations π as above with $\pi^{K_{\Sigma_0}K^{\Sigma}}$ of arbitrarily high dimension. This is done by explicit computations of possible lifts in [BD]. □

This proposition indicates that the $\widehat{H}^1_{LT,K}$ and $\widehat{H}^1_{LT,\bar{\mathbb{F}}_p}$ are non-admissible smooth representations.

8.3. **Non-admissibility.** We have

Proposition 8.9. *The* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representations* $\widehat{H}^1_{ss,A}$ *and* $\widehat{H}^2_{X_{ord},A}$ *are non-admissible smooth* A*-representations, where* A *is either* $\bar{\mathbb{F}}_p$ *or* K*.*

Proof. If one of them would be admissible, then also the second would because of the exact sequence

$$\widehat{H}^1_{X_{ord},A} \to \widehat{H}^1_A \to \widehat{H}^1_{ss,A} \to \widehat{H}^2_{X_{ord},A} \to \widehat{H}^2_A$$

But we know that $\widehat{H}^1_{X_{ord},A}$ is an induced representation

$$\mathrm{Ind}_{B(\infty)}^{\mathrm{GL}_2}\left(\bigoplus_a \widehat{H}^1_{\mathbb{X}_{a,\infty},A}\right)$$

so if it were admissible, then it would have to vanish and we would have

$$\widehat{H}^1_{ss,A,(\pi)} \simeq \widehat{H}^1_{A,(\pi)}$$

We claim that this is not possible. Observe firstly that $\widehat{H}^1_{LT,A,(\pi)}$ is an injective representation of $\mathrm{GL}_2(\mathbb{Z}_p)$. This follows from the standard argument (that we have already given for $\mathbf{F}$) and the fact that $H^0_{LT,A,(\pi)} = H^2_{LT,A,(\pi)} = 0$ under our assumptions. Take K^p-invariants (where K^p is a sufficiently small compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_f^p)$) and localise the above isomorphism at $\mathfrak{m}$ to get

$$(\widehat{H}^1_{LT,A,(\pi)} \otimes \mathbf{F}^{K^p}_{A,\mathfrak{m}})^{D^\times(\mathbb{Q}_p)} \simeq (\widehat{H}^1_{A,\mathfrak{m}})^{K^p}$$

By the proposition above we know that $\mathbf{F}^{K^p}_{A,\mathfrak{m}}$ is a $D^\times(\mathbb{Q}_p)$-representation of infinite length. This means, as $\widehat{H}^1_{LT,A,(\pi)}$ is an injective representation of $\mathrm{GL}_2(\mathbb{Z}_p)$, that also $(\widehat{H}^1_{LT,A,(\pi)} \otimes \mathbf{F}^{K^p}_{A,\mathfrak{m}})^{D^\times(\mathbb{Q}_p)}$ and hence $(\widehat{H}^1_{A,\mathfrak{m}})^{K^p}$ is of infinite length. This is a contradiction, as Emerton has proved in [Em2] that $(\widehat{H}^1_{A,\mathfrak{m}})^{K^p}$ is of finite length (actually, he has proved much more, namely he has completely described $(\widehat{H}^1_{A,\mathfrak{m}})^{K^p}$). □

Corollary 8.10. *The* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representation* $\widehat{H}^1_{LT,A}$ *is a non-admissible smooth* A*-representation.*

Proof. Follows from the proposition above. □

9. Localisation of smooth representations

We will discuss in this section how one can localise at a supersingular representation π smooth representations which are not neccessarily admissible. This will be useful in the next section to say something about non-admissibility of the representations we consider. Let us firstly introduce a notion of a locally admissible representation (see [Em3])

Definition 9.1. *A locally admissible representation is a representation which can be written as a direct limit of admissible representations.*

Observe that we can consider for locally admissible representations a localisation with respect to a supersingular represention π exactly as for admissible representations, see [Pa3]. In [EP1], Emerton and Paskunas have proved the following result (see also Corollary 5.15 in [Pa1])

Proposition 9.2. *An inclusion from the category of locally admissible representations to the category of smooth representations takes injective objects to injective objects.*

This allows us to conclude that the injective envelope of π in the category of smooth representations is equal to the injective envelope of π in the category of locally admissible representations. Let us define the category $\mathfrak{C}(\mathcal{O})$ as the full abelian subcategory of pro-augmented $\mathrm{GL}_2(\mathbb{Q}_p)$-modules closed under products and subquotients and furthermore such that every object M in $\mathfrak{C}(\mathcal{O})$ can be written as $M \simeq \varprojlim_i M_i$, where the limit is taken over all the quotients of finite length. Let $\mathfrak{C}(k)$ be the full subcategory of $\mathfrak{C}(\mathcal{O})$ consisting of objects which are killed by ϖ. We see that the Pontryagin dual $\pi^\vee$ of π is an object of $\mathfrak{C}(k)$. Now we can use the following result of Paskunas:

Proposition 9.3. *The projective envelope P of $\pi^\vee$ in $\mathfrak{C}(\mathcal{O})$ has all its irreducible subquotients and subrepresentations isomorphic to $\pi^\vee$. Moreover $P \otimes_{\mathcal{O}} k$ is the projective envelope of $\pi^\vee$ in $\mathfrak{C}(k)$ and it satisfies the same property.*

Proof. By Proposition 6.1 from [Pa1], we can use the formalism from section 3 of the same article. □

Those results allows us to apply the general theorem (see Lemma B.0.1 in [Da1]):

Theorem 9.4. *Let $\mathcal{C}$ be an abelian category with products and coproducts and let S be a set of simple objects of $\mathcal{C}$. Denote by $\mathcal{C}_S$, (respectively, $\mathcal{C}^S$) the full subcategory of $\mathcal{C}$ which consists of objects with all the simple subquotients in S (respectively, no simple subquotient in S). If $\mathcal{C}_S$ contains an object P, which is projective in $\mathcal{C}$ and such that $\mathrm{Hom}_{\mathcal{C}}(P, s) \neq 0$ for all $s \in S$, then every object M in $\mathcal{C}$ has a unique filtration $V_S \hookrightarrow V \twoheadrightarrow V^S$, with $V_S \in \mathcal{C}_S$ and $V^S \in \mathcal{C}^S$.*

We apply it to the category of pro-augmented modules over $\mathcal{O}$ and k (which is category dual to the category of smooth representations over $\mathcal{O}$ and k), $S = \{\pi^\vee\}$, P equal to the projective envelope of $\pi^\vee$ either in $\mathfrak{C}(\mathcal{O})$ or in $\mathfrak{C}(k)$ (which is the same as the projective envelope of $\pi^\vee$ in the whole category of pro-augmented modules by proposition 9.2). By dualizing it, we obtain filtrations in the category of smooth representations. For every $V \in \mathrm{Rep}^{sm}(\mathcal{O})$ (resp. $\mathrm{Rep}^{sm}(k)$) we have a unique filtration

$$V^{(\pi)} \hookrightarrow V \twoheadrightarrow V_{(\pi)}$$

with $V_{(\pi)} \in \mathrm{Rep}^{sm}(\mathcal{O})_{(\pi)}$ and $V^{(\pi)} \in \mathrm{Rep}^{sm}(\mathcal{O})^{(\pi)}$ (resp. $\mathrm{Rep}^{sm}(k)_{(\pi)}$ and $\mathrm{Rep}^{sm}(k)^{(\pi)}$), where we have denoted by $\mathrm{Rep}^{sm}(\mathcal{O})_{(\pi)}$ (resp. $\mathrm{Rep}^{sm}(\mathcal{O})^{(\pi)}$) the category of smooth $\mathcal{O}$-representations whom all irreducible subquotients are isomorphic to π (resp. no irreducible subquotient is isomorphic to π). This permits us to localise smooth representations with respect to π as we did for admissible representations.

10. Cohomology with compact support

In this section we will discuss what happens when we consider the cohomology with compact support. Our basic result is negative and it states that the first cohomology group with compact support of the fundamental representation $\widehat{H}^1_{LT,c,\bar{\mathbb{F}}_p}$ does not contain any supersingular representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ as a subrepresentation. This suprising result, which is very different from the situation known in the l-adic setting where $l \neq p$, leads to a similar exact sequence as we have considered for cohomology without support, but this time, we get that $\pi \otimes \bar{\rho}$ is contained in the H^1 of the ordinary locus.

10.1. **Geometry at pro-p Iwahori level.** Let $K(1) = \begin{pmatrix} 1+p\mathbb{Z}_p & p\mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}$ and let $I(1) = \begin{pmatrix} 1+p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}$ be the pro-p Iwahori subgroup. We let

$$\mathcal{M}_{LT,K(1)} = \mathrm{Spf}\, R_{K(1)}$$
$$\mathcal{M}_{LT,I(1)} = \mathrm{Spf}\, R_{I(1)}$$

be the formal models for the Lubin-Tate space at levels $K(1)$ and $I(1)$ respectively. We will compute $R_{I(1)}$ explicitely. This is also done in a more general setting in the work of Haines-Rapoport (see Corollary 3.4.3 in [HR]) but here we give a short and elementary argument.

We know that $R_{I(1)} = R_{K(1)}^{I(1)}$ and hence we can use the explicit description of $R_{K(1)}$ by Yoshida to get the result (see Proposition 3.5 in [Yo]). Let $W = W(\bar{\mathbb{F}}_p)$ be the Witt vectors of $\bar{\mathbb{F}}_p$. There is a surjection $W[[\tilde{X}_1, \tilde{X}_2]] \twoheadrightarrow R_{K(1)}$ which maps $\tilde{X}_i$ to X_i where X_i $(i = 1, 2)$ is a local parameter for $R_{K(1)}$. We will find parameters for $R_{I(1)} = R_{K(1)}^{I(1)}$. Observe that for $b \in \mathbb{F}_p$ we have (see chapter 3 of [Yo])

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} X_1 = X_1$$

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} X_2 = [b]X_1 +_\Sigma X_2$$

where we use the notations from chapter 3 of [Yo] for [.] and $+_\Sigma$ which are considered on the universal deformation of the unique formal group over $\bar{\mathbb{F}}_p$ of height 2. We see that X_2 is not invariant under $I(1)$ and hence we define $X_2' = \prod_{b \in \mathbb{F}_p}([b]X_1 +_\Sigma X_2)$ which is. We claim that (X_1, X_2') are local parameters for $R_{I(1)}$. Indeed if z belongs to $R_{I(1)} = R_{K(1)}^{I(1)}$ then we may write it as $z = a_0 +_\Sigma P(X_1) +_\Sigma X_2 Q(X_1, X_2)$, where $a_0 \in W$, $P \in W[[X_1]]$ and $Q \in W[[X_1, X_2]]$. As $a_0 +_\Sigma P(X_1)$ is invariant under $I(1)$, we see that also $X_2 Q(X_1, X_2)$ has to be invariant under $I(1)$. Because of the action of $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ on X_2 described above and the fact that $R_{K(1)}$ is a regular ring hence factorial, we see that X_2' divides $X_2 Q(X_1, X_2)$ (we use here the fact that $[b]X_1 +_\Sigma X_2$ and $[b']X_1 +_\Sigma X_2$ are not associated for $b \neq b'$; this follows from Proposition 4.2 in [Str]). This leads to $z = a_0 +_\Sigma P(X_1) +_\Sigma X_2' Q'(X_1, X_2)$ for some Q' and hence we conclude by induction and the fact that polynomials are dense in formal series.

Let us observe that for $a \in \mathbb{F}_p^\times$ we have for $i = 1, 2$: $[a]X_i = (\text{unit})X_i$. Let us now look at the relation defining $R_{K(1)}$ inside $W[[\tilde{X}_1, \tilde{X}_2]]$ which appears in Proposition 3.5 of [Yo]. We have

$$p = u \prod_{(a_1, a_2) \in \mathbb{F}_p^2 \setminus \{0,0\}} ([a_1]X_1 +_\Sigma [a_2]X_2)$$

where u is some unit. Thus we have modulo units

$$p \equiv \prod_{(a_1, a_2) \in \mathbb{F}_p^2 \setminus \{0,0\}} ([a_1]X_1 +_\Sigma [a_2]X_2) \equiv \left(\prod_{a_1 \in \mathbb{F}_p^\times} [a_1]X_1 \right) \left(\prod_{a_1 \in \mathbb{F}_p} \prod_{a_2 \in \mathbb{F}_p^\times} [a_2]([a_1/a_2]X_1 +_\Sigma X_2) \right) \equiv$$

$$\equiv \left(\prod_{a_1 \in \mathbb{F}_p^\times} [a_1]X_1 \right) \left(\prod_{a_2 \in \mathbb{F}_p^\times} [a_2]X_2' \right) \equiv (X_1 X_2')^{p-1}$$

Hence we have $p = u'(X_1 X_2')^{p-1}$ in $R_{I(1)}$ for some unit u'. Because $W[[X, Y]]$ is a complete ring with an algebraically closed residue field there exists a $(p-1)$-th root of u'. We want to conclude that this is the only relation in $R_{I(1)}$ which means that there exists a surjection

$$B = W[[\tilde{X}_1, \tilde{X}_2']] \twoheadrightarrow R_{I(1)}$$

with kernel $f = (XY)^{p-1} - p$. First of all, observe that $R_{I(1)}$ and B/fB are regular local rings of dimension 2 with a surjection $B/fB \twoheadrightarrow R_{I(1)}$. We claim that this map has to be neccessarily an injection also. Indeed, this holds for any surjective morphism $A \twoheadrightarrow R$ of regular local rings of the same dimension by using the fact that that for a regular local ring we have $\mathrm{gr}^\bullet_{\mathfrak{m}_A} A \simeq \mathrm{Sym}\, \mathfrak{m}_A/\mathfrak{m}_A^2$ and similarly for $\mathfrak{m}_A/\mathfrak{m}_A^k$. This yields an isomorphism at the graded level which lifts to a level of rings as k is arbitrary. All in all, we conclude that

Proposition 10.1. *We have*

$$R_{I(1)} = W[[X, Y]]/((XY)^{p-1} - p)$$

This means that $\mathcal{M}_{LT, I(1)}$ is made of $p-1$ copies of an annulus in $\mathbb{P}^1$ (after a base change to $\mathbb{Z}_p[\zeta_p]$).

10.2. **Cohomology at pro-p Iwahori level.** We compute $H^1_c(\mathcal{M}_{LT,I(1)}, \bar{\mathbb{F}}_p)$ (we will omit $\bar{\mathbb{F}}_p$ from the notation in what follows). Let $\mathcal{A}$ be an annulus in $\mathbb{P}^1$. We can write a long exact sequence

$$0 \to H^0_c(\mathcal{A}) \to H^0(\mathbb{P}^1) \to H^0(\mathbb{P}^1 \backslash \mathcal{A}) \to H^1_c(\mathcal{A}) \to H^1(\mathbb{P}^1) \to 0$$

We know that

$$H^1(\mathbb{P}^1) = H^0_c(\mathcal{A}) = 0$$
$$\dim_{\bar{\mathbb{F}}_p} H^0(\mathbb{P}^1) = 1$$
$$\dim_{\bar{\mathbb{F}}_p} H^0(\mathbb{P}^1 \backslash \mathcal{A}) = 2$$

and hence it follows that

$$\dim_{\bar{\mathbb{F}}_p} H^1_c(\mathcal{A}) = 1$$

Because geometrically $\mathcal{M}_{LT,I(1)}$ is made of $p-1$ copies of $\mathcal{A}$, we have

$$\dim_{\bar{\mathbb{F}}_p} H^1_c(\mathcal{M}_{LT,I(1)}) = p-1$$

Let $\mathcal{H} = \mathcal{H}_{\mathrm{GL}_2}(I(1)) = \bar{\mathbb{F}}_p[I(1)\backslash \mathrm{GL}_2(\mathbb{Q}_p)/I(1)]$ be the mod p Hecke algebra at the pro-p Iwahori level. Let I be the Iwahori subgroup of $GL_2(\mathbb{Z}_p)$. We look at the action of $\bar{\mathbb{F}}_p[I/I(1)] \simeq \bar{\mathbb{F}}_p[(\mathbb{F}_p^\times)^2]$ on the cohomology. We know by [Str] that it acts by determinant on connected components of $\mathcal{M}_{LT,K(1)}$ and hence on connected components of $\mathcal{M}_{LT,I(1)}$ so we have a decomposition of $H^1_c(\mathcal{M}_{LT,I(1)})$ into $p-1$ pieces of dimension 1:

$$H^1_c(\mathcal{M}_{LT,I(1)}) = \bigoplus_{\chi:\mathbb{F}_p^\times \to \bar{\mathbb{F}}_p^\times} H^1_c(\mathcal{M}_{LT,I(1)})_\chi$$

where $H^1_c(\mathcal{M}_{LT,I(1)})_\chi$ is the part of $H^1_c(\mathcal{M}_{LT,I(1)})$ on which $\bar{\mathbb{F}}_p[\mathbb{F}_p^\times]$ acts through χ.

10.3. **Vanishing result.** We will now prove that the supersingular representation π does not appear in $\widehat{H}^1_{LT,c,\bar{\mathbb{F}}_p}$. First of all, remark that it is enough to prove that the $\mathcal{H}$-module $\pi^{I(1)}$ does not appear in $(\widehat{H}^1_{LT,c,\bar{\mathbb{F}}_p})^{I(1)}$, because the functor $\pi \mapsto \pi^{I(1)}$ induces a bijection between supersingular representations and supersingular Hecke modules (see [Vi3]). We have the Hochschild-Serre spectral sequence (see Appendix A)

$$H^i(I(1), \widehat{H}^j_{LT,c,\bar{\mathbb{F}}_p}) \Rightarrow H^{i+j}_{LT,c,I(1),\bar{\mathbb{F}}_p}$$

where we have denoted by $H^{i+j}_{LT,c,I(1),\bar{\mathbb{F}}_p}$ the fundamental representation at $I(1)$-level. This gives a long exact sequence

$$0 \to H^1(I(1), \widehat{H}^0_{LT,c,\bar{\mathbb{F}}_p}) \to H^1_{LT,c,I(1),\bar{\mathbb{F}}_p} \to (\widehat{H}^1_{LT,c,\bar{\mathbb{F}}_p})^{I(1)} \to H^2(I(1), \widehat{H}^0_{LT,c,\bar{\mathbb{F}}_p})$$

Because $\widehat{H}^0_{LT,c,\bar{\mathbb{F}}_p} = 0$ as $H^0_c(\mathcal{M}_{LT}, \bar{\mathbb{F}}_p) = 0$ we have an $\mathcal{H}$-equivariant isomorphism

$$H^1_{LT,c,I(1),\bar{\mathbb{F}}_p} \to (\widehat{H}^1_{LT,c,\bar{\mathbb{F}}_p})^{I(1)}$$

This means that if $\pi^{I(1)}$ appears in $(\widehat{H}^1_{LT,c,\bar{\mathbb{F}}_p})^{I(1)}$ then it appears also in $H^1_{LT,c,I(1),\bar{\mathbb{F}}_p}$. But because $H^1_{LT,c,I(1),\bar{\mathbb{F}}_p}$ consists of multiple copies of $H^1_c(\mathcal{M}_{LT,I(1)}, \bar{\mathbb{F}}_p)$, it is enough to show that $\pi^{I(1)}$ does not appear in $H^1_c(\mathcal{M}_{LT,I(1)}, \bar{\mathbb{F}}_p)$. We know that $\pi^{I(1)}$ is a non-semisimple two-dimensional $\mathcal{H}$-module (see Corollary 4.1.4 in [Br1]). Because $H^1_{LT,c,I(1),\bar{\mathbb{F}}_p}$ consists of multiple copies of $H^1_c(\mathcal{M}_{LT,I(1)}, \bar{\mathbb{F}}_p)$, it is enough to show that $\pi^{I(1)}$ does not appear in $H^1_c(\mathcal{M}_{LT,I(1)}, \bar{\mathbb{F}}_p)$. To see it, observe that if $\pi^{I(1)}$ would appear in $H^1_c(\mathcal{M}_{LT,I(1)}, \bar{\mathbb{F}}_p)$ then $I/I(1)$ would act by determinant on $\pi^{I(1)}$ by preceding section which is not the case as one can see from the description given in [Br1] (see the proof of Proposition 4.1.2 in [Br1]). All in all, this means that $\pi^{I(1)}$ does not appear in $(H^1_{LT,c,\bar{\mathbb{F}}_p})^{I(1)}$ and hence

Theorem 10.2. *The supersingular representation π does not appear in $\widehat{H}^1_{LT,c,\bar{\mathbb{F}}_p}$.*

We could rephrase it also as

$$\widehat{H}^1_{LT,c,\bar{\mathbb{F}}_p,(\pi)} = 0$$

This gives us, when combined with the exact sequence for the supersingular locus, an appearance of the mod p local Langlands correspondence in the cohomology of the ordinary locus

Corollary 10.3. *We have an* $\mathrm{GL}_2(\mathbb{Q}_p) \times G_{\mathbb{Q}_p}$*-equivariant injection*

$$\pi \otimes \bar{\rho} \hookrightarrow \widehat{H}^1_{ord,\bar{\mathbb{F}}_p,(\pi)}$$

Moreover, this vanishing result can be used in the study of non-admissibility and in the description of the cohomology of certain Shimura curves.

10.4. **Non-admissibility.** We will now show that our cohomology groups are non-admissible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. We start with:

Proposition 10.4. *The* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representations* $\widehat{H}^2_{ss,c,A}$ *and* $\widehat{H}^1_{ord,A}$ *are non-admissible smooth A-representations, where A is either* $\bar{\mathbb{F}}_p$ *or* K*.*

Proof. If one of them would be admissible, then also the second would because of the exact sequence

$$\widehat{H}^1_{ss,c,A} \to \widehat{H}^1_A \to \widehat{H}^1_{ord,A} \to \widehat{H}^2_{ss,c,A} \to \widehat{H}^2_A$$

But we know that $\widehat{H}^1_{ord,A}$ is an induced representation

$$\mathrm{Ind}^{\mathrm{GL}_2}_{B(\infty)} \left(\bigoplus_a \widehat{H}^1_{a,\infty,A} \right)$$

so if it were admissible, then it would have to vanish. □

Corollary 10.5. *The* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representation* $\widehat{H}^2_{LT,c,A}$ *is a non-admissible smooth A-representation.*

Proof. Follows from the proposition above. □

Actually we can go even deeper into the structure of the above representations and consider everything at the localised level. For this, remark firstly that after the localisation we will still have an exact sequence

$$0 \to \widehat{H}^1_{A,(\pi)} \to \widehat{H}^1_{ord,A,(\pi)} \to \widehat{H}^2_{ss,c,A,(\pi)} \to 0$$

which comes from the fact that $\widehat{H}^1_A$ is admissible and hence the localisation exact sequence from section 9

$$0 \to (\widehat{H}^1_A)^{(\pi)} \to \widehat{H}^1_A \to \widehat{H}^1_{A,(\pi)} \to 0$$

is split. Recall now the Borel subgroup $B := B(\infty)$ from Section 2. Paskunas has proved (see [Pa4]) that a supersingular representation π of $\mathrm{GL}_2(\mathbb{Q}_p)$ when restricted to B rests irreducible. Moreover, Berger and Vienney (see [BV]) have recently classified irreducible $\bar{\mathbb{F}}_p$-representations of B and they have proved that they all arise by a certain construction of Colmez used in the p-adic Langlands correspondence. This lead us to a hope that we can localise smooth B-representations just as we did for representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ above. We will work below under the assumption that such a localisation indeed exists and is compatible with the localisation for $\mathrm{GL}_2(\mathbb{Q}_p)$. Then

Proposition 10.6. *The* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representations* $\widehat{H}^2_{ss,c,A,(\pi)}$ *and* $\widehat{H}^1_{ord,A,(\pi)}$ *are non-admissible smooth A-representations, where A is either* $\bar{\mathbb{F}}_p$ *or* K*.*

Proof. If one of them would be admissible, then also the second would because of the exact sequence

$$0 \to \widehat{H}^1_{A,(\pi)} \to \widehat{H}^1_{ord,A,(\pi)} \to \widehat{H}^2_{ss,c,A,(\pi)} \to 0$$

But we know that $\widehat{H}^1_{ord,A,(\pi)}$ is an induced representation

$$\mathrm{Ind}^{\mathrm{GL}_2}_{B(\infty)} \left(\bigoplus_a \widehat{H}^1_{a,\infty,A,(\pi_{|B})} \right)$$

where $(\pi_{|B})$ denotes the localisation at the restriction of the supersingular representation π. If $\widehat{H}^1_{ord,A,(\pi)}$ were admissible, then it would have to vanish. $\square$

Corollary 10.7. *The* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representation* $\widehat{H}^2_{LT,c,A,(\pi)}$ *is a non-admissible smooth* A*-representation.*

Proof. Follows from the proposition above. $\square$

10.5. **Cohomology of Shimura curves.** We will briefly sketch another consequence of vanishing of $\widehat{H}^1_{LT,c,\bar{\mathbb{F}}_p,(\pi)}$. First of all, it implies that $H^1_c(\mathcal{M}_{LT},\bar{\mathbb{F}}_p)_{(\pi)}$ vanishes because $\widehat{H}^1_{LT,c,\bar{\mathbb{F}}_p,(\pi)}$ is just a sum of copies of $H^1_c(\mathcal{M}_{LT})_{(\pi)}$. Now recall the Faltings isomorphism (see [Fa2]) which gives us

$$H^1_c(\mathcal{M}_{LT},\bar{\mathbb{F}}_p)_{(\pi)} = H^1_c(\mathcal{M}_{Dr},\bar{\mathbb{F}}_p)_{(\pi)} = 0$$

where $\mathcal{M}_{Dr}$ is the Drinfeld tower at infinity (see [Da1] for details). We have a spectral sequence coming from the p-adic uniformisation of the Shimura curve Sh associated to the algebraic group G'' arising from the quaternion algebra over $\mathbb{Q}$ which is ramified precisely at p and some other prime q:

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathrm{GL}_2(\mathbb{Q}_p)}(H^{2-q}_c(\mathcal{M}_{Dr,K_p},\bar{\mathbb{F}}_p), C^\infty(G'(\mathbb{Q})\backslash G'(\mathbb{A}),\bar{\mathbb{F}}_p)^{K^p}) \Rightarrow H^{p+q}_c(Sh^{an}_{K_pK^p},\bar{\mathbb{F}}_p)$$

where we have denoted by G' the algebraic group arising from the quaternion algebra over $\mathbb{Q}$ which is ramified precisely at q and ∞. For this, see [Fa1] where it is proven for $\bar{\mathbb{Q}}_l$ but the proof works also for $\bar{\mathbb{F}}_p$ (the proof is also contained in the appendix B of [Da2]) .

Choose any non-Eisenstein maximal ideal $\mathfrak{n}$ in the Hecke algebra of G'' whose associated Galois representation corresponds at p to the supersingular representation π we have chosen before. Take the direct limit over K_p and localise the above spectral sequence at $\mathfrak{n}$ to get

$$\mathrm{Ext}^p_{\mathrm{GL}_2(\mathbb{Q}_p)}(H^{2-q}_c(\mathcal{M}_{Dr},\bar{\mathbb{F}}_p)_{(\pi)}, C^\infty(G'(\mathbb{Q})\backslash G'(\mathbb{A}),\bar{\mathbb{F}}_p)^{K^p}_{\mathfrak{n}}) \Rightarrow H^{p+q}_c(Sh^{an}_{K^p},\bar{\mathbb{F}}_p)_{\mathfrak{n}}$$

The localisation of $H^{2-q}_c(\mathcal{M}_{Dr},\bar{\mathbb{F}}_p)$ at π appears because $C^\infty(G'(\mathbb{Q})\backslash G'(\mathbb{A}),\bar{\mathbb{F}}_p)^{K^p}_{\mathfrak{n}}$ is π-isotypic. Using our vanishing result we get an interesting isomorphism

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(H^2_c(\mathcal{M}_{Dr},\bar{\mathbb{F}}_p)_{(\pi)}, C^\infty(G'(\mathbb{Q})\backslash G'(\mathbb{A}),\bar{\mathbb{F}}_p)^{K^p}_{\mathfrak{n}}) \simeq H^1_c(Sh^{an}_{K^p},\bar{\mathbb{F}}_p)_{\mathfrak{n}}$$

This can be possibly used to study the mod p cohomology of the Shimura curve Sh. We shall treat this issue elsewhere.

11. p-adic non-abelian Lubin-Tate theory

At the end, we will shortly discuss what happens in the p-adic setting. Due to either geometric or formal nature of our results, most results still hold. We are still able to compare the cohomology of the Lubin-Tate tower with that of modular curves and we can define a candidate for the p-adic Jacquet-Langlands correspondence, but our results are weaker than those obtained in the mod p setting. It is of independent interest to try to define the p-adic Jacquet-Langlands correspondence by purely local means, which would give a first instance of the p-adic Langlands correspondence behind the $\mathrm{GL}_2(\mathbb{Q}_p)$-case. Details for this costruction shall appear elsewhere.

Recall that we have obtained an isomorphism

$$\widehat{H}^1_{ss,K} \simeq \left(\widehat{\mathbf{F}} \otimes_K \widehat{H}^1_{LT,K}\right)^{D^\times(\mathbb{Q}_p)}$$

and also an injection

$$(\widehat{H}^1_K)_{(\pi)} \hookrightarrow \widehat{H}^1_{ss,K}$$

hence

$$(\widehat{H}^1_K)_{(\pi)} \hookrightarrow \left(\widehat{\mathbf{F}} \otimes_K \widehat{H}^1_{LT,K}\right)^{D^\times(\mathbb{Q}_p)}$$

Let $\bar{\rho}$ be a globalisation of a local representation $\bar{\rho}_p$ which is modular with the associated Hecke ideal $\mathfrak{m}$ and which is unramified outside $\Sigma = \Sigma_0 \cup \{p\}$ and let ρ be a pro-modular lift of $\bar{\rho}$, so there exists an ideal $\mathfrak{p} \in \mathrm{Spec}\, \mathbb{T}_{\mathfrak{m},\Sigma}$ associated to ρ. Let us denote by $K_{\mathfrak{p}} = \prod_{l\neq p} K_{\mathfrak{p},l}$ a compact open subgroup of $\prod_{l\neq p}\mathrm{GL}_2(\mathbb{Z}_l)$ such that $\pi_l(\rho)^{K_{\mathfrak{p},l}}$ is one-dimensional for every $l\neq p$. Let us take in the above injection $K_{\mathfrak{p}}$-invariants and $[\mathfrak{p}]$-isotypic part getting

$$(\widehat{H}^1_K)^{K_{\mathfrak{p}}}_{(\pi)}[\mathfrak{p}] \hookrightarrow \left(\widehat{\mathbf{F}}^{K_{\mathfrak{p}}}[\mathfrak{p}]\otimes_K \widehat{H}^1_{LT,K}\right)^{D^\times(\mathbb{Q}_p)}$$

Put

$$\widehat{\sigma}_{\mathfrak{p}} = \widehat{\mathbf{F}}^{K_{\mathfrak{p}}}[\mathfrak{p}]$$

Theorem 11.1. *We have a* $\mathrm{GL}_2(\mathbb{Q}_p)\times G_{\mathbb{Q}_p}$*-equivariant injection*

$$B(\rho_p)\otimes_K \rho_p \hookrightarrow \widehat{H}^1_{LT,K}[\widehat{\sigma}_{\mathfrak{p}}^\vee]$$

Proof. This follows from the injection written above and theorem 7.3:

$$(\widehat{H}^1_K)^{K_{\mathfrak{p}}}_{(\pi)}[\mathfrak{p}] \simeq B(\rho_p)\otimes_K \rho_p$$

□

The $D^\times(\mathbb{Q}_p)$-representation $\widehat{\sigma}_{\mathfrak{p}}$ is a natural candidate for the p-adic Jacquet-Langlands correspondence. The following result describes a bit of its structure:

Proposition 11.2. *For any open, bounded* $D^\times(\mathbb{Q}_p)$*-invariant lattice* θ *in* $\widehat{\sigma}_{\mathfrak{p}}^{l.alg}$*, all the irreducible subquotients of* $\theta\otimes_{\mathcal{O}}\bar{\mathbb{F}}_p$ *are isomorphic to* $(\mathrm{Ind}\,\alpha)^\vee$*. Here we have denoted by l.alg the locally algebraic vectors in* $\widehat{\sigma}_{\mathfrak{p}}$*.*

Proof. It is enough to prove it just for one such lattice. From Proposition 8.5 follows that the only irreducible representations appearing in $\mathbf{F}^{K_{\mathfrak{p}}}[\mathfrak{p}]$ are isomorphic to $(\mathrm{Ind}\,\alpha)^\vee$. By Lemma 7.4.1 in [EGH] we have

$$\widehat{\mathbf{F}}^{K_{\mathfrak{p}}}_{\mathcal{O}}[\mathfrak{p}]^{l.alg}\otimes_{\mathcal{O}}\bar{\mathbb{F}}_p \simeq \mathbf{F}^{K_{\mathfrak{p}}}[\mathfrak{p}]$$

and hence the result. □

It is natural to ask whether we have a local-global compatibility result in the p-adic setting

Conjecture 11.3. *We have a* $D^\times(\mathbb{A}_f)$*-equivariant isomorphism*

$$\widehat{\mathbf{F}}[\mathfrak{p}] \simeq \widehat{\mathbf{F}}^{K_{\mathfrak{p}}}[\mathfrak{p}]\otimes_K \pi^p(\rho)$$

Another natural question is whether $\widehat{\sigma}_{\mathfrak{p}}$ depends on $\mathfrak{p}$ (we conjecture that it does not). Let us remark that similar facts might also be considered at the integral level, which is natural in light of global results of Emerton in [Em2] and a structure of considered objects which are deformations of certain mod p objects. Again we remark, that the above conjecture would be true if one could prove the existence of a Colmez functor with sufficiently good properties in the setting of quaternion algebras.

12. Concluding remarks

Let us finish by giving some remarks and stating natural questions.

12.1. **l-adic case.** Observe that our arguments work well also in the mod $l\neq p$ setting and circumvent the use of vanishing cycles. An idea of localisation at a supersingular (supercuspidal) representation appears also in the work of Dat. See especially [Da1] where the author discusses localisations both for GL_n and quaternion algebras and then uses it to describe the supercuspidal part of the cohomology.

One might want also to see [Sh], which bears some resemblance to certain arguments we use. Shin describes a mod l cohomology of Shimura varieties by using results of Dat about the mod l cohomology of the Lubin-Tate tower. In our work, we start from global results of Emerton to deduce from them statements about local objects.

12.2. **Beyond modular curves.** The geometric arguments we have given also applies to Shimura curves considered by Carayol in [Ca] and we can consider similar exact sequences relating the ordinary locus and the supersingular locus in this setting. Nevertheless, in this case we cannot go on with arguments as we do not have a definition of the mod p local Langlands correspondence for extensions of $\mathbb{Q}_p$. In fact, such a construction seems a little bit problematic as might be seen from the work of Breuil-Paskunas ([BP]), where the authors show that there are much more automorphic representations than Galois representations. The hope is that by looking at the cohomology of the Lubin-Tate tower, one should be able to tell how the correspondence should look like. We will pursue this subject in our subsequent work.

12.3. **Ordinary locus.** In the section 2, we have introduced a decomposition of the ordinary locus of modular curves. Is this decomposition, a decomposition into irreducible components? For a similar question in a little bit different setting, see [Col].

12.4. **Adic spaces.** We have chosen to work with Berkovich spaces, but one might as well wonder how the things translate into the setting of adic spaces of R. Huber ([Hu]). In fact, everything that we have considered can be rewritten in the language of adic spaces and we might consider the same long exact sequences as above. The difference between those two contexts lies in the ordinary locus which in the case of adic spaces will contain additional points which lie in the closure of the ordinary locus from the setting of Berkovich spaces. Nevertheless, the cohomology groups will agree in both these settings. Let us remark also, that the comparison between mod p étale cohomology of a formal scheme and its (adic) analytification is proved in Theorem 3.7.2 of [Hu].

We thank Peter Scholze for clarifying this point to us.

12.5. **Jacquet-Langlands correspondence.** We have already mentioned following questions in our text: is it possible to give more intrinsic characterisation of the mod p Jacquet-Langlands correspondence than that appearing in our definition, which would be defined purely in terms of a local representation and would not refer to a global lift? Are local representations $\sigma_{\mathfrak{m}}$ independent of $\mathfrak{m}$? We can pose similar questions in the p-adic setting.

12.6. **Serre's letters.** Though it does not appear explicitely in our work (besides the comparison of Hecke algebras), we were influenced by two letters written by Jean-Pierre Serre (see [Se]). It is there that in some sense appears for the first time the modified mod l Local Langlands correspondence which goes under the name of the universal unramified representation (see the letter to Kazhdan). Indeed, if we were to suppose that our global lift $\bar{\rho}$ which we have used is actually unramified everywhere outside p, then there is no need to recall either the modified mod l Local Langlands correspondence or new vectors, and we could formulate everything in the language of Serre.

Appendix A. The Hochschild-Serre spectral sequence

In the body of the text we have used the unpublished manuscript of Berkovich ([Ber4]) where, among others, appears the Hochschild-Serre spectral sequence for the cohomology with compact support. For the sake of completeness, we will sketch a proof of existence of this spectral sequence here. We thank Vladimir Berkovich for sending us his preprint.

A.1. **G-spaces.** Recall that an analytic space (in the sense of Berkovich) is a k-analytic space over some non-archimedean field k. Given two analytic spaces X and Y, let $\mathrm{Mor}(X, Y)$ denote the set of morphisms $X \to Y$ and let $\mathcal{G}(X)$ be the group of automorphisms of X. Berkovich defined in [Ber2] a uniform space structure (and in particular, a topology) on $\mathrm{Mor}(X, Y)$. Then, the group $\mathcal{G}(X)$ has the topology induced from $\mathrm{Mor}(X, X)$. We say that the action of a topological group G on an analytic space X is continuous if the induced homomorphism $G \to \mathcal{G}(X)$ is continuous. An analytic space endowed with a continuous action of a topological group G will be called a G-space. A G-equivariant morphism between two G-spaces will be called a G-morphism. The category of analytic spaces is the category of pairs (X, G), where G is a topological group and X is a G-space. We will denote this pair by $X(G)$. A morphism between such spaces $\phi : X'(G') \to X(G)$ is a pair consisting of a continuous homomorphism of topological groups $\nu_\phi : G' \to G$ and a morphism of analytic spaces $\phi : X' \to X$

compatible with the homomorphism ν_ϕ. A G-morphism $\phi : X' \to X$ between G-spaces gives rise to a morphism $\phi : X'(G) \to X(G)$ for which ν_ϕ is the identity map on G. If X is a G-space then the action of G on X extends to a natural action of G on $X(G)$ for which $\nu_g(g') = gg'g^{-1}$, where ν_g is the morphism given by an element $g \in G$. For a G-space X we have a morphism $b : X \to X(G)$ where $X = X(\{1\})$.

A.2. **Étale topology.** Berkovich has defined the étale topology on analytic spaces and similarly we can define the étale topology on G-analytic spaces. For a G-space X, let $Et(X(G))$ denote the category of étale morphisms $U(G) \to X(G)$. The étale topology on $X(G)$ is the Grothendieck topology on the category $Et(X(G))$ with coverings of $U(G) \to X(G)$ consisting of families $(U_i(G) \to U(G))_{i\in I}$ such that $(U_i \to U)_{i\in I}$ is a covering in the étale topology of X. We denote this site by $X(G)_{et}$ and its corresponding topos by $X(G)^{\sim}_{et}$. In a similar way, we can also introduce a quasi-étale site $X(G)_{qet}$ and its topos.

We denote by $\Gamma_{X(G)}$ the functor of global sections on $X(G)_{et}$, that is $\Gamma_{X(G)}(F) = F(X(G))$. The higher direct images of $\Gamma_{X(G)}$ on the category of abelian sheaves will be denoted by $F \mapsto H^i(X(G), F)$. Let F be an étale abelian sheaf on $X(G)$. The support of $f \in F(X(G))$ is the (closed) set $\mathrm{Supp}(f) = \{x \in X \mid f_x \neq 0\}$, where f_x is the image of f in F_x. The cohomology groups with compact support are higher direct images of the functor $F \mapsto \Gamma_{c,X(G)}(F) = \{f \in F(X(G)) \mid \mathrm{Supp}(f) \text{ is compact}\}$ and are denoted by $F \mapsto H^i_c(X(G), F)$. We consider also the higher direct functor of $F \mapsto \Gamma_{c,X\{G\}}(F) := \varinjlim \Gamma_{c,X(N)}(F)$ where N runs through open subgroups of G which we will denote by $F \mapsto H^i_c(X\{G\}, F)$. We have $H^i_c(X\{G\}, F) = \varinjlim H^i_c(X(N), F)$. The proposition (Corollary 1.5.2 in [Ber4]) we have used in the main text was

Proposition A.1. *For any étale abelian sheaf F on $X(G)$ there are canonical isomorphisms*

$$H^i_c(X\{G\}, F) \simeq H^i_c(X, b^*F)$$

for $i \geq 0$. In particular, there is a spectral sequence

$$E_2^{p,q} = H^p(G, H^q_c(X, b^*F)) \Rightarrow H^{p+q}_c(X(G), F)$$

We have applied it to $X = \mathcal{M}_{LT}$ the Lubin-Tate tower at infinity and $G = I(1)$ the pro-p Iwahori subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$, using also the fact that we have an equivalence of topoi $X(G)^{\sim}_{et} \simeq (G\backslash X)^{\sim}_{et}$, whenever $G\backslash X$ exists and $X \to G\backslash X$ is étale. This is so in our case, where $G\backslash X = \mathcal{M}_{LT,I(1)}$.

Proof. We sketch the proof of this proposition. The case $i = 0$ follows from the fact that every element of $H^0_c(X, b^*F)$ is fixed by an open subgroup of G. Then the general case follows by constructing the Godement resolution in our context. Namely, for a topological space I denote by $Top(I)$ the site on the category of local homeomorphisms $J \to I$ (with the evident topology). Suppose we have a surjective map $I \to X : i \mapsto x_i$. We endow I with the discrete topology and we fix for each $i \in I$ a geometric point $\bar{x}_i$ over x_i. This gives rise to a morphism of sites $\nu : Top(I) \to X(G)_{et}$. For an étale abelian sheaf F on $X(G)$, its Godement resolution $\mathcal{C}^\bullet(F)$ is constructed as follows:

(i) $\mathcal{C}^0(F) = \nu_*\nu^*(F)$ and let $d^{-1} : F \to \mathcal{C}^0(F)$ be the adjunction map

(ii) if $m \geq 0$, then put $\mathcal{C}^{m+1}(F) = \mathcal{C}^0(coker\ d^{m-1})$, and let d^m be the canonical map $\mathcal{C}^m(F) \to \mathcal{C}^{m+1}(F)$.

This construction is taken from SGA 4, Exp. XVII, where it is shown that

(a) $\mathcal{C}^m(F)$ is a flabby sheaf

(b) the functor $F \mapsto \mathcal{C}^m(F)$ is exact

(c) the fibre of the complex $\mathcal{C}^\bullet(F)$ at a point $x \in X$ is a canonically split resolution of F_x.

Then Berkovich shows (Proposition 1.5.1), that for any $F \in X(G)^{\sim}_{et}$ and $m \geq 0$, the sheaf $b^*(\mathcal{C}^m(F))$ is soft on X_{et}, where we say that a sheaf $\mathcal{F}$ is soft on X_{et} (after chapter 3 in [Ber2]) when it satisfies the following two conditions

(1) for any $x \in X$, the stalk $\mathcal{F}_x$ is a flabby $\mathrm{Gal}_{\mathcal{H}(x)} = \mathrm{Gal}(\overline{\mathcal{H}(x)}/\mathcal{H}(x))$-module, where $\mathcal{H}(x)$ is the complete field associated to x by the standard procedure.

(2) for any paracompact U étale over X, the restriction of $\mathcal{F}$ to the usual topology $|U|$ of U is a soft sheaf, that is, for any compact subset $T \subset U$, the map $\mathcal{F}(U) \to \mathcal{F}(T)$ is surjective.

Then one shows (see Lemma 3.2 in [Ber2]) that we can compute the cohomology with soft sheaves on X_{et}. Remark also that to prove the theorem it is enough to prove that $\mathcal{F} = b^*(\mathcal{C}^0(F))$ is soft, by the inductive definition of the Godement resolution. This is done by checking explicitely the conditions (1) and (2) for such an $\mathcal{F}$. We omit the computations. □

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