FROM CLASSIC HODGE THEORY TO P-ADIC

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This is a quick overview of recent results of Fargues and Fontaine from [**FF1**] (see also [**FF2**] for the survey). We try also to explain a motivation behind it. These notes are still a preliminary draft. Any comments and suggestions are welcome.

1. Classic Hodge theory

Let X be a complex manifold of dimension n. We have the following holomorphic de Rham complex

$$0 \to \mathcal{O}_X \to \Omega^1_X \to ... \to \Omega^n_X \to 0$$

and by (holomorphic) Poincare's lemma this complex is a resolution of the constant sheaf $\mathbb{C}$ on X. Thus

$$H^n_{dR} := \mathbb{H}^n(X, \Omega^\bullet_X) \simeq H^n(X, \mathbb{C})$$

where $\mathbb{H}^n(X, \Omega^\bullet_X)$ is the hypercohomology of the complex $\Omega^\bullet_X$ on X and $H^n(X, \mathbb{C})$ denotes singular cohomology. This amounts to an existence of the spectral sequence

$$E_2^{p,q} = H^p(X, \Omega^q_X) \Rightarrow H^{p+q}(X, \mathbb{C})$$

Let us pose $H^{p,q}(X) = H^p(X, \Omega^q_X)$. This is also a set of closed (p,q) forms. By a Kahler manifold we mean a complex manifold which admits a Hermitian form h which can be described in local parameters as $h = \sum h_{i,j} dz_i \otimes d\bar{z}_j$ and whose associated real form $\omega = \frac{i}{2} \sum h_{i,j} dz_i \wedge d\bar{z}_j$ is closed. Hodge theory gives us:

Proposition 1.1. — *Let X be a compact Kahler manifold, then the above spectral sequence degenerates, that is, we have a decomposition*

$$H^m(X, \mathbb{C}) = \bigoplus_{p+q=m} H^{p,q}(X)$$

We show one application of this result. We will cite facts from [**BHPV**].

Definition 1.2. — A K3 surface is compact complex surface X with trivial canonical divisor $K_X = 0$ and for which $b_1(X) = dim H^1(X, \mathbb{C}) = 0$.

An example of a K3 surface is quartic hypersurface on $\mathbb{P}^3$ given by $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$. We know that K3 surfaces are Kahler manifolds hence we can write a decomposition

$$H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

We introduce a lattice $L = (-E_8)^2 \oplus H^3$, where $H = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and E_8 is the intersection matrix of E_8 Dynkin diagram. Then L is free $\mathbb{Z}$-module of rank 22 and it admits a bilinear form $(\cdot,\cdot)_L$. One can prove the following theorem:

Theorem 1.3. — *We have $H^1(X, \mathbb{Z}) = H^3(X, \mathbb{Z}) = 0$ and $H^2(X, \mathbb{Z})$ is a torsion free module of rank 22 which is isometric to L, where the bilinear form on $H^2(X, \mathbb{Z})$ is the one induced by the cup-product $H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to \mathbb{Z}$. Moreover, in the Hodge decomposition above, we have $dim H^{2,0}(X) = dim H^{0,2}(X) = 1$ and $dim H^{1,1}(X) = 20$.*

Definition 1.4. — A marked K3 surface is a K3 surface X together with an isometry

$$\phi : H^2(X, \mathbb{Z}) \to L$$

We introduce the following period domain

$$\Omega = \{[w] \in \mathbb{P}(L \otimes \mathbb{C}) | (w, w)_L = 0, (w, \bar{w})_L > 0\}$$

where $\mathbb{P}(L \otimes \mathbb{C})$ denotes the set of lines in complexification of L. Let us denote by ω_X the real form coming from Kahler metric on X. It lies in $H^{1,1}(X)$. One can prove that for an isometry $\phi : H^2(X, \mathbb{Z}) \to L$, $\phi(\omega_X) \in \Omega$. The weak Torelli theorem tells

Theorem 1.5. — *For every $x \in \Omega$ there exists a marked K3 surface (X, ϕ) such that $x = \phi(\omega_X)$. Moreover two K3 surfaces X, X' are isomorphic if and only if there exists an isometry $\phi : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ which preserves Hodge decomposition.*

This tells us that the period domain Ω is a coarse moduli space of marked K3 surfaces.

2. B_{dR}-conjecture

From now on, we will work in the algebraic setting. For a scheme X, its de Rham cohomology $H^n_{dR}(X)$ is defined in the same way, by using hypercohomology of (algebraic) differential forms, i.e.

$$H^n_{dR}(X) = \mathbb{H}^n(X, \Omega^{\bullet}_X)$$

Let X be a proper, smooth scheme over $\mathbb{C}$. In the same vein as in the section 1, we have

$$H^n_{dR}(X) \simeq H^n(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

This isomorphism is obtained by integrating differential forms from H_{dR} over cycles in the singular homology. The result of an integration is called a period and the result above tells us that $\mathbb{C}$ is a period ring (this terminology will be clearer later on).

Let us denote by $\mathbb{Q}_p$ the p-adic numbers, $\bar{\mathbb{Q}}_p$ its algebraic closure, and let X be a proper, smooth scheme over $\mathbb{Q}_p$. Then the B_{dr}-conjecture of Fontaine (which is now a theorem) states:

$$B_{dR} \otimes_{\mathbb{Q}_p} H^n_{dR}(X) \simeq H^n_{et}(X \times_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{dR}$$

and this is an isomorphism of filtered spaces with $G_{\mathbb{Q}_p} := Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$-action. Here, B_{dR} is one of the period rings of Fontaine (p-adic periods) and H^n_{et} denotes the etale cohomology. There are also other period rings which include B_{cris} and B_{st} related to crystalline and log-crystalline cohomology. The B_{dR}-conjecture might also be stated as:

$$D_{dR}(H^n_{et}(X \times_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p, \mathbb{Q}_p)) := (B_{dR} \otimes_{\mathbb{Q}_p} H^n_{et}(X \times_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p, \mathbb{Q}_p))^{G_{\mathbb{Q}_p}} \simeq H^n_{dR}(X)$$

The etale cohomology group $H^n_{et}(X \times_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p, \mathbb{Q}_p)$ is an example of p-adic Galois representation. Galois representations are the main object of study of p-adic Hodge theory.

3. p-adic Galois representations

3.1. Definitions. — Let $G_{\mathbb{Q}_p} = Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) = \varprojlim_{K/\mathbb{Q} \text{ finite}} Gal(K/\mathbb{Q}_p)$. We put a discrete topology on $Gal(K/\mathbb{Q}_p)$ for each finite extension $K/\mathbb{Q}_p$. Then, we put the topology of an inverse limit on $G_{\mathbb{Q}_p}$, that is, the weakest topology for which $G_{\mathbb{Q}_p} \to Gal(K/\mathbb{Q}_p)$ are continuous for each finite extension $K/\mathbb{Q}_p$. This is so-called Krull topology.

Definition 3.1. — A p-adic Galois representation is a continuous representation $\rho : G_{\mathbb{Q}_p} \to GL_(V)$ where V is a finite dimensional $\mathbb{Q}_p$-vector space, and where $GL(V)$ is considered with the p-adic topology and $G_{\mathbb{Q}_p}$ with the Krull topology.

We now give a definition of the Fontaine's ring B_{dR}. Let $\mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$ be the completion with respect to a p-adic norm of the algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Let $\mathcal{O}_{\mathbb{C}_p} = \{x \in \mathbb{C}_p | |x|_p \leq 1\}$ be its ring of integers. We define:

$$R = \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/(p) = \{(x_n)_{n \geq 0} \in (\mathcal{O}_{\mathbb{C}_p}/(p))^{\mathbb{N}} | (x^p_{n+1} = x_n\}$$

This is a perfect $\mathbb{F}_p$-algebra. Moreover its fraction field $Frac(R)$ is algebraically closed. We have a map

$$\theta_0 : R \to \mathcal{O}_{\mathbb{C}_p}/(p)$$

defined as a projection $(x_n)_n \mapsto x_0$. We would like to "lift" it to

$$\theta : W(R) \to \mathcal{O}_{\mathbb{C}_p}$$

where $W(R)$ is the ring of Witt vectors of R, which is defined $W(R) = \{\sum_{k\geq 0} r_k p^n | r_k \in R\} = R^{\mathbb{N}}$ as a set. We can define θ via $\theta(\sum_k r_k p^n) = \sum r_k^{(0)} p^n$. Let us also denote by the same letter the extension of θ to

$$\theta : W(R)\left[\frac{1}{p}\right] \to \mathcal{O}_{\mathbb{C}_p}\left[\frac{1}{p}\right] = \mathbb{C}_p$$

Then we define

$$B_{dR}^{+} = \varprojlim_{j} W(R)\left[\frac{1}{p}\right]/(ker\theta)^j$$

and we put

$$B_{dR} = Frac(B_{dR}^{+})$$

Observe that there is a Galois action of $G_{\mathbb{Q}_p}$ on B_{dR} which comes from a Galois action on R. There is also a filtration on B_{dR} by the $\mathbb{Z}$-powers of the maximal ideal of B_{dR}^{+}. One can prove

Lemma 3.2. —

$$B_{dR}^{G_{\mathbb{Q}_p}} = \mathbb{Q}_p$$

Hence we can consider a functor

$$V \mapsto D_{dR}(V) = (B_{dR} \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}$$

which already appeared when we have talked about B_{dR}-conjecture.

Definition 3.3. — We say that a p-adic Galois representation V is de Rham if $dim_{\mathbb{Q}_p} D_{dR}(V) = dim_{\mathbb{Q}_p} V$. Analogously V is (log)-crystalline if the similar equality of dimensions hold for $D_{cris}(V)$, $D_{st}(V)$, where those functors are defined in the same way replacing B_{dR} by B_{cris} or B_{st}.

3.2. Equivalence of categories. — We will now explain an equivalence of category of p-adic Galois representations with the category of (ϕ,Γ)-modules. Let $\mathbb{A}^+ = W(R)$ and $\mathbb{A} = W(FracR)$. There is an action of $G_{\mathbb{Q}_p}$ and an operator ϕ on $\mathbb{A}^+$ given by

$$\phi\left(\sum_{k=0}^{\infty}[x_k]p^k\right) = \sum_{k=0}^{\infty}[x_k^p]p^k$$

hence there is also an action of these two things on other rings we consider.

Choose $\varepsilon = (1, \varepsilon^{(1)}, \varepsilon^{(2)}, ...) \in R$ such that $\varepsilon^{(1)} \neq 1$, i.e. this is a compatible system of primitive roots of 1. It plays the role of $e^{2\pi i}$. Now, let $\pi = [\varepsilon] - 1 \in W(R) = \mathbb{A}^+$, where $[\varepsilon]$ is a Teichmuller lift of ε, that is $[\varepsilon] = (\varepsilon, 0, 0, ...)$ in $W(R)$. Then define

$$\mathbb{A}_{\mathbb{Q}_p}^{+} = \mathbb{Z}_p[[\pi]] \hookrightarrow \mathbb{A}^+$$

and

$$\mathbb{A}_{\mathbb{Q}_p} = \widehat{\mathbb{Z}_p[[\pi]][\frac{1}{\pi}]} \hookrightarrow \mathbb{A}$$

where we have denoted by hat the completion with respect to p-adic topology, thus

$$A_{\mathbb{Q}_p} = \{\sum_{k\in\mathbb{Z}} a_k \pi^{-k} | a_k \in \mathbb{Z}_p, v_p(a_k) \to \infty \text{ when } k \to -\infty\}$$

Put $B_{\mathbb{Q}_p} = Frac(\mathbb{A}_{\mathbb{Q}_p}) = A_{\mathbb{Q}_p}[\frac{1}{p}]$. Define also a cyclotomic extension $K_\infty = \bigcup_{n\in\mathbb{N}} K_n$ where $K_n = \mathbb{Q}_p(\varepsilon^{(n)})$ and $H = Gal(\bar{\mathbb{Q}}_p/K_\infty)$, $\Gamma = G_{\mathbb{Q}_p}/H = Gal(K_\infty/\mathbb{Q}_p)$.

Definition 3.4. — A (ϕ,Γ)-module over $B_{\mathbb{Q}_p}$ is a finite dimensional $B_{\mathbb{Q}_p}$-vectore space with semi-linear continuous and commuting actions of ϕ and Γ.

A (ϕ,Γ)-module is etale (or of slope 0) if there exists a basis $\{e_1, ..., e_d\}$ of it over $B_{\mathbb{Q}_p}$ such that the matrix of $\phi(e_1), ..., \phi(e_d)$ in the basis $e_1, ..., e_d$ is inside $GL_d(\mathbb{A}_{\mathbb{Q}_p})$.

***Theorem 3.5* (Fontaine).** — *The functor*

$$V \mapsto D(V) = (B_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} V)^H$$

induces an equivalence of (tensor)-categories from the category of p-adic Galois representations to the category of etale (ϕ, Γ)-modules over $B_{\mathbb{Q}_p}$.

This theorem permits to study Galois representations by using some linear-algebraic objects. Later on, we will identify semistable sheaves of slope 0 on the Fargues-Fontaine curve with (ϕ, Γ)-modules of slope 0, hence with the p-adic Galois representations.

4. The Fargues-Fontaine curve

We introduce norms on $\mathbb{A}^+ = W(R)$ for any $r > 0$ by $|| \sum_{n \gg -\infty} [x_n] p^n ||_r = sup\{|x_n| p^{-rn}\}$. Let $\mathbb{A}_r^+ = \{x \in \mathbb{A}^+ | ||x||_r \leq 1\}$ and $\mathbb{B}_r^+ = \mathbb{A}_r^+ \left[\frac{1}{p}\right]$ and $\mathbb{B}^+ = \bigcap_{r>0} \mathbb{B}_r^+$. Observe that on $\mathbb{B}^+$ there is still defined an operator ϕ. We introduce graded algebras

$$P_h = \bigoplus_{d \geq 0} (\mathbb{B}^+)^{\phi^h = p^d}$$

for any $h \geq 1$. We also introduce curves

$$X_h = Proj P_h$$

This is the main object of study. For $h = 1$, we will simply write $P = P_1$ and $X = X_1$ for the Fargues-Fontaine curve.

***Theorem 4.1* (Fargues-Fontaine).** — *$(X_h)_{h \geq 1}$ is a generalized Riemann sphere.*

Definition 4.2. — A generalized Riemann sphere $(X_h)_{h \geq 1}$ consists of a complete curve $X_1 = X$ and the tower of etale Galois coverings $(X_h)_{h \geq 2}$ of X with the Galois group $\widehat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$ i.e. we have projections $\pi_h : X_h \to X$ and $\pi_{h',h} : X_{h'} \to X_h$ such that the Galois groups of X_h over X is $\mathbb{Z}/h\mathbb{Z}$ and the Galois group of $X_{h'}$ over X_h is $h\mathbb{Z}/h'\mathbb{Z}$. Moreover, we demand the following properties to hold:

1) For each $h \geq 1$ there exists a point $\infty_h \in X_h$ (of degree 1) such that $X_h \backslash \{\infty_h\} = Spec(B)$ is affine and $Pic(X_h \backslash \{\infty_h\}) = 0$, i.e. B is principal.
2) For each $h \geq 1$, $H^1(X_h, \mathcal{O}_h) = 0$
3) For each $h \geq 1$ and each $x \in X_h$, $\pi_h^{-1} = \{x_1, ..., x_h\}$ where x_i are pairwise distinct.
4) For $h'|h$, $(\pi_{h',h})_* \mathcal{O}_{X_{h'}} \simeq \mathcal{O}_{X_h}^{h'\mathbb{Z}/h\mathbb{Z}}$.

Let us define for $d \in \mathbb{Z}$ and $h \in \mathbb{Z}_{\geq 1}$

$$\mathcal{O}_X(d, h) = \pi_{h_*}(\mathcal{O}_{X_h}(d))$$

where $\mathcal{O}_{X_h}(d) = \widetilde{P_h[d]}$. For $\lambda \in \mathbb{Q}$ we put $\mathcal{O}_X(\lambda) = \mathcal{O}_X(d, h)$, where d, h are chosen so that $(d, h) = 1$ and $h > 0$. One proves that vector bundles $\mathcal{O}_X(\lambda)$ are semistable of slope λ. Moreover we have

Theorem 4.3. — *There is a bijection:*

$$\{\textit{isomorphism classes of vector bundles on } X\} \longleftrightarrow \{(\lambda_i)_{1 \leq i \leq r} \in \mathbb{Q}^n | n \in \mathbb{N}, \lambda_1 \geq ... \lambda_n\}$$

given via $(\lambda_1, ..., \lambda_n) \mapsto \bigoplus_{i=1}^n \mathcal{O}_X(\lambda_i)$.

From this we get that the category of vector bundles on X is equivalent to the category of ϕ-modules on $\mathbb{B}^+$, where by a ϕ-module we mean a finitely generated $\mathbb{B}^+$-module M together with an isomorphism $\phi_M : M \simeq M$ which is ϕ-linear. This starts to resemble the equivalence of (ϕ, Γ)-modules with Galois representations. We also get a bijection:

$$\{\text{ semistable vector bundles of slope 0 on } X\} \longleftrightarrow \{\text{finite dimensional } \mathbb{Q}_p\text{-vector spaces}\}$$

via $\mathcal{F} \mapsto H^0(X, \mathcal{F})$. Recall that there is an action of $G_{\mathbb{Q}_p}$ on $\mathbb{B}^+$, hence on P, hence on the curve X. We introduce a category of $G_{\mathbb{Q}_p}$-equivariant vector bundles on X, those are bundles with an action of

$G_{\mathbb{Q}_p}$ which is compatible with an action of $G_{\mathbb{Q}_p}$ on X and moreover which is continuous (we do not define this notion here). Then one obtains an equivalence of categories

$$\{\text{semistable } G_{\mathbb{Q}_p}\text{-equivariant sheaves of slope 0 on} X\} \simeq \{\text{p-adic Galois representations}\}$$

via $\mathcal{F} \mapsto H^0(X, \mathcal{F})$, where the continuous Galois action on $H^0(X, \mathcal{F})$, comes from the action on $\mathcal{F}$.

In this setting we can also define what is a de Rham (resp. crystalline, log-crystalline) vector bundle and then get a bijection

$$\{\text{semistable de Rham sheaves of slope 0 on} X\} \simeq \{\text{de Rham Galois representations}\}$$

and similarly for crystalline and log-crystalline representations. This interpretation permitted Fargues and Fontaine to reprove classic results of p-adic Hodge theory using geometrical tools.

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