AN ADJUNCTION FORMULA FOR THE EMERTON–JACQUET FUNCTOR

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Abstract. The Emerton–Jacquet functor is a tool for studying locally analytic representations of p-adic Lie groups. It provides a way to access the theory of p-adic automorphic forms. Here we give an adjunction formula for the Emerton–Jacquet functor, relating it directly to locally analytic inductions, under a strict hypothesis that we call *non-critical*. We also further study the relationship to socles of principal series in the non-critical setting.

Contents

1. Introduction

Let p be a prime. Throughout this paper we fix a finite extension $K/\mathbf{Q}_p$ and let $\mathbf{G}$ be a connected, reductive and split algebraic group defined over K. We fix a maximal split torus $\mathbf{T}$ and a Borel subgroup $\mathbf{B}$ containing $\mathbf{T}$. We also let $\mathbf{P}$ denote a standard parabolic containing $\mathbf{B}$. The parabolic subgroup $\mathbf{P}$ has a Levi decomposition $\mathbf{P} = \mathbf{N_P L_P}$, with $\mathbf{N_P}$ being the maximal unipotent subgroup of $\mathbf{P}$. We denote the opposite subgroup of $\mathbf{P}$ by $\mathbf{P}^-$. If $\mathbf{H}$ is an algebraic group defined over K we use the Roman letters $H := \mathbf{H}(K)$ to denote the corresponding p-adic Lie group. Let L be another finite extension of $\mathbf{Q}_p$, which we will use as a field of coefficients.

1.1. **The Emerton-Jacquet functor.** If V is a smooth and admissible L-linear representation of G then one can associate a smooth and admissible L-linear representation $J_P(V) := V_{N_P}$ of the Levi factor L_P, called the Jacquet module of V. The functor $J_P(-)$ is exact, and it turns out that the irreducible constituents of $J_P(V)$ give rise to irreducible constituents of V via adjunction with smooth parabolic induction $\mathrm{Ind}_{P^-}^G(-)^{\mathrm{sm}}$. Specifically, if U is a smooth and admissible representation of L_P, seen as a representation of P^- via inflation, then there is a natural isomorphism

$$\mathrm{Hom}_G(\mathrm{Ind}_{P^-}^G(U)^{\mathrm{sm}}, V) \xrightarrow{\simeq} \mathrm{Hom}_{L_P}(U(\delta_P), J_P(V)), \tag{1}$$

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where δ_P is the modulus character of P.

In [Eme2], Emerton extended the functor $J_P(-)$ to certain categories of locally analytic representations of G on L-vector spaces. We will call this extended functor the Emerton–Jacquet functor but still denote it by $J_P(-)$. If U is a suitable locally analytic representation of L_P and V is a suitable locally analytic representation of G, one could ask for an adjunction formula in the spirit of (1), relating $\mathrm{Hom}_{L_P}(U(\delta_P), J_P(V))$ to $\mathrm{Hom}_G(\mathrm{Ind}_{P^-}^G(U)^{\mathrm{an}}, V)$. Here, $\mathrm{Ind}_{P^-}^G(-)^{\mathrm{an}}$ is the locally analytic induction and suitable refers to, for example, the hypotheses in the introduction of [Eme3].

It was explained by Emerton that the naïve generalization of (1) is not generally correct. Indeed, the main result of Emerton's paper [Eme3] is an isomorphism

$$\mathrm{Hom}_G(I_{P^-}^G(U), V) \xrightarrow{\simeq} \mathrm{Hom}_{L_P}(U(\delta_P), J_P(V))^{\mathrm{bal}} \tag{2}$$

where $I_{P^-}^G(U)$ is a certain *subrepresentation* of $\mathrm{Ind}_{P^-}^G(U)^{\mathrm{an}}$ and $\mathrm{Hom}_{L_P}(U(\delta_P), J_P(V))^{\mathrm{bal}}$ is the "balanced" subspace of $\mathrm{Hom}_{L_P}(U(\delta_P), J_P(V))$ (see [Eme3, Definition 0.8]).

Breuil showed in [Bre2] that one can remove the balanced condition on the right-hand side of Emerton's formula (2) at the expense of replacing the locally analytic induction $\mathrm{Ind}_{P^-}^G(U)^{\mathrm{an}}$, or its subrepresentation $I_{P^-}^G(U)$, with a closely related locally analytic representation defined and studied by Orlik and Strauch [OS1] (we recall work of Orlik–Strauch and Breuil in Section 5).

1.2. **Statement of theorem.** Our main goal is to give sufficient, practical, conditions under which a naïve adjunction formula holds. Let us state our main theorem, and comment on the hypotheses afterwards. Below, German fraktur letters refer to Lie algebras. See Section 1.6 for all the notations.

Theorem A (Theorem 4.9). *Suppose that V is a very strongly admissible locally analytic representation of G which is $\mathfrak{p}$-acyclic, U is a finite-dimensional locally analytic representation of L_P which is irreducible as a module over $\mathrm{U}(\mathfrak{l}_P)$ and π is a smooth representation of L_P which admits a central character. Then, if (U, π) is non-critical with respect to V, there is a canonical isomorphism*

$$\mathrm{Hom}_G(\mathrm{Ind}_{P^-}^G(U \otimes \pi(\delta_P^{-1}))^{\mathrm{an}}, V) \xrightarrow{\simeq} \mathrm{Hom}_{L_P}(U \otimes \pi, J_P(V)).$$

The first hypothesis, that V is very strongly admissible, is a relatively natural one. For example, it appears in [Eme3, Theorem 0.13]. We refer to Definition 4.1 for this, and for the acyclicity hypothesis. Let's mention two concrete examples we have in mind. The first is the space $V = \mathcal{C}^{\mathrm{an}}(G^0, L)$ of locally analytic L-valued functions on a compact open subgroup $G^0 \subset G$, which satisfies the stronger condition that V is actually $\mathfrak{g}$-acyclic [ST2, Proposition 3.1]. The second example is the localization of p-adically completed (compactly supported) cohomology of modular curves at non-Eisenstein ideals (see [BE, Corollarie 5.1.3]). The former arises in the theory of overconvergent p-adic modular forms and the latter models p-adic automorphic forms on definite unitary groups (unitary groups which are compact at infinity).

The hypotheses on U and π taken individually should be self-explanatory. The crucial hypothesis in Theorem A then is that the pair (U, π) be *non-critical* with respect to V. We note immediately that there is a sufficient condition, the condition of having *non-critical slope*, which depends only on U and π (see Remark 4.5) but the definition of non-critical is more general and depends on V. We will give a brief explanation here and refer to Definition 4.2 for more details.

Suppose for the moment that $\mathbf{P} = \mathbf{B}$ is a Borel subgroup so that $\mathbf{L}_P = \mathbf{T}$ is the torus. The hypothesis on U in Theorem A implies that U is a locally analytic character χ of T, and we assume that π is trivial (by absorbing π into χ). If χ' is a locally analytic character of T then there is the notion of χ' being strongly linked to χ (see Appendix A) which generalizes the well-known notion of strongly linked weights coming from the representation theory of the Lie algebra of G (see [Hum]). The Emerton–Jacquet module $J_P(V)$ for V as in Theorem A has a locally analytic action of T. We say that the character χ (or the pair $(\chi, 1)$) is non-critical with respect to V if:

1) The eigenspace $J_P(V)^{T=\chi}$ is non-zero and
2) $J_P(V)^{T=\chi'} = (0)$ for every character $\chi' \neq \chi$ strongly linked to χ.

The definition is inspired by the definition of "not bad" that appears in [BE] and [BC].

For general $\mathbf{P}$, the definition of non-critical is more complicated. Even if $\mathbf{P}$ is not a Borel subgroup, the hypothesis on U still allows us to obtain a character χ of T, a "highest weight" for U, which we may restrict to the center $Z(L_P)$ of the Levi factor. The definition of non-critical is then phrased in terms of the eigenspaces of $J_P(V)$ under the action of the center $Z(L_P)$. The subtlety of which characters to qualify over is already present in the intricacy of so-called *generalized* Verma modules [Hum, Section 9]. However, we suggest in Section 1.5 why one should consider characters of the center, and one possible strategy to checking the non-critical hypothesis.

1.3. **Relationship with locally analytic socles.** Theorem A is meant to capture a strong relationship between representations appearing in an Emerton–Jacquet module $J_P(V)$ and principal series appearing in V. However, in contrast with the classical theory of smooth and admissible representations, the principal series $\mathrm{Ind}_{P^-}^G(U \otimes \pi)^{\mathrm{an}}$ may be highly reducible. For example, it is easy to construct examples of U, π and V such that every non-zero map $\mathrm{Ind}_{P^-}^G(U \otimes \pi)^{\mathrm{an}} \to V$ factors through a proper quotient. And so, for Theorem A to be useful, one needs to prove a stronger version. For that, we restrict to the case where U is an irreducible finite-dimensional *algebraic* representation of L_P, i.e. the induced representation of L_P on an irreducible finite-dimensional algebraic representation of the underlying algebraic group $\mathbf{L}_P$.

Theorem B (Theorem 5.4)**.** *Suppose that U is an irreducible finite-dimensional algebraic representation of L_P and π is a finite length smooth representation of L_P admitting a central character, such that $\mathrm{Ind}_{P^-}^G(\pi)^{\mathrm{sm}}$ is irreducible. Let V be a very strongly admissible, $\mathfrak{p}$-acyclic representation of G such that (U, π) is non-critical with respect to V. Then the containment $\mathrm{soc}_G \mathrm{Ind}_{P^-}^G(U \otimes \pi(\delta_P^{-1}))^{\mathrm{an}} \subset \mathrm{Ind}_{P^-}^G(U \otimes \pi(\delta_P^{-1}))^{\mathrm{an}}$ induces a natural isomorphism*

$$\mathrm{Hom}_G(\mathrm{Ind}_{P^-}^G(U \otimes \pi(\delta_P^{-1}))^{\mathrm{an}}, V) \simeq \mathrm{Hom}_G(\mathrm{soc}_G \mathrm{Ind}_{P^-}^G(U \otimes \pi(\delta_P^{-1}))^{\mathrm{an}}, V)$$

Here, the notation $\mathrm{soc}_G(-)$ refers to the locally analytic *socle*, i.e. the sum of the topologically irreducible subrepresentations (under these assumptions, it is actually irreducible). The hypothesis on π is sufficient, but not necessary (see the statement of Theorem 5.4 and Remark 5.6). The socles of principal series play a central role in recent conjectures of Breuil [Bre2]. Combining Theorems A and B results in an obvious corollary, which we omit.

1.4. **Methods.** The proof of Theorem A comes in two steps. The first step is to understand composition series of certain $(\mathfrak{g}, P)$-modules, which generalize the composition series of Verma modules. This is carried out in Sections 2 and 3 and constitutes a significant portion of our work. The second step is to use the description of the composition series to prove Theorem A. This is done in Section 4. The crux of the argument follows from the difficult p-adic functional analysis studied by Emerton in his two papers on the Emerton–Jacquet functor [Eme2, Eme3].

The proof of Theorem B, which we give in Section 5, naturally falls out of the techniques we use to prove Theorem A, together with the progress made by Orlik and Strauch [OS1, OS2], and Breuil [Bre1], on understanding locally analytic principal series (a suitable generalizations of some results in [Bre1] would possibly remove the algebraic hypothesis on U; see Remark 5.5).

1.5. **Global motivation.** From a purely representation-theoretic point of view, one might wonder why the definition of non-critical qualifies over characters of the center, rather than representations of the Levi subgroup. In order to explain this, let us finish the introduction by placing our theorem and its hypotheses within the still emerging p-adic Langlands program.

A rich source of locally analytic representations of p-adic Lie groups comes from the theory of p-adically completed cohomology. See [CE] for a summary and further references. In the theory, one constructs p-adic Banach spaces, typically denoted by $\widehat{H}^i$, arising from the p-adically completed cohomology of locally symmetric spaces for an adelic group which locally at a place above p is the p-adic Lie group G. The completion process equips $\widehat{H}^i$ with a natural continuous action of G. Passing to locally analytic vectors $\widehat{H}^i_{\mathrm{an}}$, we get a very strongly admissible locally analytic representation of G. The cohomology $\widehat{H}^i$ also has an auxiliary action of a Hecke algebra which we denote by $\mathcal{H}$. After localizing at a so-called non-Eisenstein maximal ideal $\mathfrak{m} \subset \mathcal{H}$, we get a representation $V = \widehat{H}^i_{\mathrm{an},\mathfrak{m}}$ which will satisfy the hypotheses of Theorem A (in practice—the localization here is meant to force the cohomology to be acyclic for the derived action of the Lie algebra).

Already in the case of degree zero cohomology, there has been a significant amount of energy spent predicting the principal series representations of G which should appear in spaces $\widehat{H}^0_{\mathrm{an},\mathfrak{m}}$ attached to certain definite unitary groups. See [BH, Section 4] and [Bre2, Section 6]. More precisely, [Bre2, Conjectures 6.1 and 6.2] predicts the principal series whose *socle* appears as a subrepresentation of certain cohomology spaces and [BH, Conjecture 4.2.2] gives a prediction for which continuous principal series (which arise by unitary completion from our setting) should appear. When the non-critical hypothesis is satisfied, Theorems A and B illuminates those conjectures.

The remaining link between Theorem A and p-adic Langlands is provided by the theory of *eigenvarieties*. Emerton showed in [Eme1], see also the work of Loeffler and Hill [HL], how to use the Emerton–Jacquet module $J_P(\widehat{H}^i_{\mathrm{an}})$ to construct a p-adic rigid analytic space $\mathcal{E}$ called an eigenvariety. The eigenvariety $\mathcal{E}$ parameterizes pairs $(\chi, \mathfrak{m})$, where χ is a locally analytic character of the center $Z(L_P)$ of the Levi and $\mathfrak{m} \subset \mathcal{H}$ is a maximal ideal, such that the χ-eigenspace for the action of T on $J_P(\widehat{H}^i_{\mathrm{an},\mathfrak{m}})$ is non-zero. The eigenvarieties constructed this way vastly generalize the p-adic *eigencurve* constructed by Coleman and Mazur [CM].

Taking $V = \widehat{H}^i_{\mathrm{an},\mathfrak{m}}$, the obstruction to applying Theorem A to V can be reduced to the existence or non-existence of certain points on the eigenvariety $\mathcal{E}$. This final question belongs in the theory of *p-adic companion forms*, a topic studied previously by the authors in [BC], which provides a test case for the motivation laid out here. For more on constructing companion forms, see [BHS, Di].

1.6. **Notations.** We specify further our notations. We fix an algebraic closure $\overline{\mathbf{Q}}_p$ and a finite extension $L/\mathbf{Q}_p$ contained in $\overline{\mathbf{Q}}_p$. Let K be another finite extension of $\mathbf{Q}_p$ and we assume throughout that $\sigma(K) \subset L$ for all embedding $\sigma : K \to \overline{\mathbf{Q}}_p$. We let bold letters $\mathbf{G}$ $\mathbf{P}$, etc. denote algebraic groups over K and Roman letters G, P, etc. for K-points. We also use German letters $\mathfrak{g}, \mathfrak{p}$, etc. for the corresponding Lie algebras over $\mathbf{Q}_p$.

If V is a $\mathbf{Q}_p$-vector space then we define $V_L := V \otimes_{\mathbf{Q}_p} L$. If V is an L-vector space we use V^* to denote the L-linear dual. Each Lie algebra $\mathfrak{g}$ has a universal enveloping algebra which we denote by $\mathrm{U}(\mathfrak{g})$ and we note that $\mathrm{U}(\mathfrak{g})_L \simeq \mathrm{U}(\mathfrak{g}_L)$. We prefer the first notation since $\mathrm{U}((\mathfrak{l}_P)_L)$ is a bit much.

The dual space $\mathfrak{t}_L^* = \mathrm{Hom}_L(\mathfrak{t}_L, L)$ is the space of weights of $\mathfrak{t}_L$. It contains the set of roots Φ, which are the weights of the adjoint action of $\mathfrak{t}$ on $\mathfrak{g}$. Our choice of the Borel subgroup $\mathbf{B} \subset \mathbf{G}$ defines a set of positive roots Φ^+ generated by a set of simple roots we denote by Δ throughout. The half-sum of the positive roots is $\rho_0 := \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha \in \mathfrak{t}_L^*$. If $\lambda \in \mathfrak{t}_L^*$ is a weight and $\eta : L \to \mathfrak{t}_L$ is a co-character then we let $\langle \lambda, \eta\rangle$ denote the canonical element of L given by $(\eta \circ \lambda)(1)$. If $\alpha \in \Phi$ is a root then there is a co-character, called the co-root of α, denote by $\alpha^\vee$. If $\chi : T \to L^\times$ is a locally analytic character of T then $d\chi \in \mathfrak{t}_L^*$ is its derivative.

If G is a group, $H \subset G$ is a subgroup and V is an L-linear representation of H then we denote by $\mathrm{Ind}_H^G(V)$ the L-vector space

$$\mathrm{Ind}_H^G(V) = \{f : G \to V \mid f(hg) = h \cdot f(g) \text{ for all } h \in H\}.$$

This is equipped with a left action of G via the right regular representation. Note that if $H = P$ is a parabolic subgroup of a connected reductive group G, this is *non-normalized* induction.

Each parabolic subgroup P determines a modulus character which we denote by $\delta_P : T \to K^\times$. If $\mathbf{G} = \mathrm{GL}_{n/\mathbf{Q}_p}$ and $P = B$ is the *upper* triangular Borel, we have for $t = \mathrm{diag}(t_1, \ldots, t_n) \in T$

$$\delta_B(t) = \prod_{i=1}^{n} |t_i|_p^{n+1-2i}.$$

For each parabolic $P \supset B$ we also fix throughout the choice of a compact open subgroup $N_P^0 \subset N_P$. If $\mathbf{G} = \mathrm{GL}_{n/K}$ then a natural choice would be to choose $N_P^0 = \mathbf{N}_P(\mathcal{O}_K)$. If V is a representation of N_P then we define an action of the monoid

$$L_P^+ = \{t \in L_P : tN_P^0 t^{-1} \subset N_P^0\} \tag{3}$$

on $V^{N_P^0}$ by

$$\pi_t \cdot v := \frac{1}{(N_P^0 : tN_P^0 t^{-1})} \sum_{a \in N_P^0/tN_P^0 t^{-1}} at \cdot v = \delta_P(t) \sum_{a \in N_P^0/tN_P^0 t^{-1}} at \cdot v, \tag{4}$$

This action appears in [Eme2, §3.3] and [Bre2, §4], though Breuil's normalization is off from ours by the appearance of the modulus character.

2. Reminder on Verma modules

Let W be an L-linear finite dimensional representation of the universal enveloping algebra $\mathrm{U}(\mathfrak{l}_P)_L$. By inflation it defines a left $\mathrm{U}(\mathfrak{p})_L$-module. We define the *generalized Verma module*

$$M_{\mathfrak{p}}(W) := \mathrm{U}(\mathfrak{g})_L \otimes_{\mathrm{U}(\mathfrak{p})_L} W.$$

Note that we could (and will) also use $M_{\mathfrak{p}^-}(W)$. The $\mathrm{U}(\mathfrak{g})_L$-module $M_{\mathfrak{p}}(W)$ lies in the full subcategory $\mathcal{O}^{\mathfrak{p}}$ consisting of left $\mathrm{U}(\mathfrak{g})_L$-modules M which satisfy the following properties (see [Hum] for more details):

- M is finite-type over $\mathrm{U}(\mathfrak{g})_L$.
- As a $\mathrm{U}(\mathfrak{l}_P)_L$-module, M is a weight module and each weight space is finite-dimensional.
- $\mathrm{U}(\mathfrak{n}_P)_L$ acts locally finitely on M, i.e. for each vector v we have $\mathrm{U}(\mathfrak{n}_P)_L \cdot v$ is a finite-dimensional L-vector space.

If $\mathbf{P} \supset \mathbf{Q}$ are two choices of parabolic subgroups then $\mathcal{O}^{\mathfrak{p}}$ is a full subcategory of $\mathcal{O}^{\mathfrak{q}}$. In particular, $\mathcal{O}^{\mathfrak{b}} \supset \mathcal{O}^{\mathfrak{p}}$ for all standard parabolics $\mathbf{P}$.

Every standard parabolic $\mathbf{P}$ is determined by a subset $I_{\mathbf{P}} \subset \Delta$ of simple roots. The extreme cases are $I_{\mathbf{B}} = \emptyset$ and $I_{\mathbf{G}} = \Delta$. Now consider the set of weights

$$\Lambda_P^+ = \{\lambda \in \mathfrak{t}_L^* \mid \langle \lambda + \rho_0, \alpha^\vee \rangle \in \mathbf{Z}_{>0} \text{ for all } \alpha \in I_{\mathbf{P}}\}.$$

The elements $\lambda \in \Lambda_P^+$ parameterize finite-dimensional irreducible $\mathrm{U}(\mathfrak{l}_P)_L$-modules; we denote this correspondence by $\lambda \leftrightarrow W_\lambda$. In the case where $\mathbf{P} = \mathbf{B}$, every λ is in Λ_B^+ and W_λ is just the weight λ itself (see Proposition 2.5 for a useful description of W_λ valid for all $\mathbf{P}$). The salient facts are given by the following.

Proposition 2.1. *Suppose that $\lambda \in \mathfrak{t}_L^*$.*

1. *The Verma module $M_{\mathfrak{b}}(\lambda)$ has a unique irreducible quotient $L(\lambda)$.*
2. *Every irreducible object in $\mathcal{O}^{\mathfrak{p}}$ is of the form $L(\lambda)$ for some $\lambda \in \Lambda_P^+$.*
3. *If $\lambda \in \Lambda_P^+$ then there is a unique $\mathrm{U}(\mathfrak{g})_L$-module quotient map $M_{\mathfrak{b}}(\lambda) \twoheadrightarrow M_{\mathfrak{p}}(W_\lambda)$ and $L(\lambda)$ is the unique irreducible quotient of $M_{\mathfrak{p}}(W_\lambda)$.*
4. *If $\lambda \in \Lambda_P^+$ then the irreducible constituent $L(\lambda)$ of $M_{\mathfrak{p}}(W_\lambda)$ appears exactly once.*

Proof. This is all well-known so we just provide references. Point (1) follows from $M_{\mathfrak{b}}(\lambda)$ being a highest weight module (see [Hum, Theorem 1.2(f)]). The point (2) follows from [Hum, Theorem 1.3] and [Hum, Proposition 9.3]. Point (3) is deduced from (1) and (2) (see [Hum, Section 9.4]). The final point, point (4), reduces to the case $\mathfrak{b} = \mathfrak{p}$ by (3). In that case, it is a consequence of highest weight theory. □

Recall that if M is an L-vector space then M^* denotes its L-linear dual. If M has an action of $\mathrm{U}(\mathfrak{g})_L$ then there are two separate actions we may endow M^* with. To uniformly explain this, suppose that $s : \mathrm{U}(\mathfrak{g})_L \to \mathrm{U}(\mathfrak{g})_L$ is an L-linear anti-involution. In that case we can define a *left* action of $\mathrm{U}(\mathfrak{g})_L$ on M^* by

$$(X \cdot_s f)(m) := f(s(X)m)$$

for $X \in \mathrm{U}(\mathfrak{g})_L$, $f \in M^*$ and $m \in M$. If M is a left $\mathrm{U}(\mathfrak{g})_L$-module we use $M^{*,s}$ to denote M^* endowed with its action via the involution s. The two dual actions on left $\mathrm{U}(\mathfrak{g})_L$-modules we will consider are obtained by taking for s one the following two choices:

- the unique anti-involution $\iota : \mathrm{U}(\mathfrak{g})_L \to \mathrm{U}(\mathfrak{g})_L$ which acts by -1 on $\mathfrak{g}_L$ (see [Bre2, §3.1]), or
- the unique anti-involution $\tau : \mathrm{U}(\mathfrak{g})_L \to \mathrm{U}(\mathfrak{g})_L$ which fixes $\mathfrak{t}_L$ point-by-point and defines an isomorphism $\tau : \mathfrak{g}_\alpha \simeq \mathfrak{g}_{-\alpha}$ for every non-zero root α (see [Hum, §0.5]).

Now suppose that in addition to a left $\mathrm{U}(\mathfrak{g})_L$-module structure, M is also a weight module for the action of $\mathfrak{t}_L$. For each weight $\lambda \in \mathfrak{t}_L^*$ let M_λ be the weight subspace of M, which is a direct summand of M. So, we consider the surjection $M \twoheadrightarrow M_\lambda$ and the corresponding L-linear inclusion $j_\lambda : (M_\lambda)^* \hookrightarrow M^*$. It is elementary to check that for each $\lambda \in \mathfrak{t}_L^*$,

$$(M^{*,\tau})_\lambda = j_\lambda((M_\lambda)^*) = (M^{*,\iota})_{-\lambda}.$$

If we define, just as an L-vector space, $M^{*,\infty} \subset M^*$ by

$$M^{*,\infty} = \bigoplus_{\lambda \in \mathfrak{t}_L^*} j_\lambda((M_\lambda)^*)$$

then we have natural vector space identifications

$$(M^{*,\iota})^{\mathfrak{n}_{B^-}^\infty} = M^{*,\infty} = (M^{*,\tau})^{\mathfrak{n}_B^\infty}, \tag{5}$$

where $V^{\mathfrak{n}_B^\infty}$ is the subspace of vectors in V annihilated by $\mathfrak{n}_B^k$ for some $k \geq 0$; similarly for $\mathfrak{n}_{B^-}^\infty$.

Definition 2.2. *Suppose that $M \in \mathcal{O}^{\mathfrak{b}}$.*

(1) *Its internal dual, denoted $M^\vee$, is the left $\mathrm{U}(\mathfrak{g})_L$-module $M^\vee := (M^{*,\tau})^{\mathfrak{n}_B^\infty}$.*
(2) *Its opposite dual, denoted by M^-, is the left $\mathrm{U}(\mathfrak{g})_L$-module $M^- := (M^{*,\iota})^{\mathfrak{n}_{B^-}^\infty}$.*

It must be checked what category the separate duals lie in. Note that since $\mathcal{O}^{\mathfrak{b}} \supset \mathcal{O}^{\mathfrak{p}}$, the dualities are defined on, but *a priori* may not preserve, the category $\mathcal{O}^{\mathfrak{p}}$.

Lemma 2.3. *Suppose that $M \in \mathcal{O}^{\mathfrak{p}}$.*

(1) *The internal dual $M^\vee$ lies in $\mathcal{O}^{\mathfrak{p}}$.*
(2) *The opposite dual M^- lies in $\mathcal{O}^{\mathfrak{p}^-}$.*
(3) *Both dualities are exact, contravariant, equivalences of categories.*
(4) *We have a commutation relation $(M^\vee)^- \simeq (M^-)^\vee$ in $\mathcal{O}^{\mathfrak{p}^-}$.*
(5) *If W is a finite-dimensional L-linear representation of $\mathfrak{l}_P$ and $\mathfrak{t}_L$ acts semi-simply on W then $(M_{\mathfrak{p}}(W)^\vee)^- \simeq M_{\mathfrak{p}^-}(W^{*,\iota})$.*

Proof. The underlying vector space of either duality is given by $M^{*,\infty} \subset M^*$. Since $\mathfrak{n}_P \subset \mathfrak{n}_B$ (and the same with the opposites substituted), the first two points are clear by definition. Furthermore, it also shows that $(M^\vee)^\vee \simeq M$ and $(M^-)^- \simeq M$, making (3) clear. To prove (4) it suffices, by (3) and Proposition 2.1, to consider $M = L(\lambda)$ where $\lambda \in \Lambda_P^+$. But by [Hum, Theorem 3.2(c)] we have $L(\lambda)^\vee = L(\lambda)$ and since $L(\lambda)^-$ and $L(-\lambda)$ are both irreducible of highest weight $-\lambda$ it is also clear that $L(\lambda)^- \simeq L(-\lambda)$. For (5), the assumption on the action of $\mathfrak{t}_L$ implies that W is completely reducible and thus we may assume that W is irreducible. In that case, it is a short computation (for example, see the unique non-numbered equation in the proof of [Bre2, Théorème 4.3]). □

Remark 2.4. One can verify the hypothesis in (5) in applications. For example, see Section 3.5.

Before moving on, let us remind the reader of the following useful formula. Here we apply the theory of Verma modules to the reductive Lie algebra $\mathfrak{l}_P$ with its Borel $\mathfrak{b}(\mathfrak{l}_P) = \mathfrak{b} \cap \mathfrak{l}_P$. This still contains the torus $\mathfrak{t}$ and so any weight $\lambda \in \mathfrak{t}_L^*$ defines a character of $\mathfrak{b}(\mathfrak{l}_P)_L$.

Proposition 2.5. *If $\lambda \in \Lambda_P^+$ then there exists a canonical sequence of $\mathrm{U}(\mathfrak{l}_P)_L$-modules*

$$\bigoplus_{\alpha \in I_P} \left(\mathrm{U}(\mathfrak{l}_P)_L \otimes_{\mathrm{U}(\mathfrak{b}(\mathfrak{l}_P))_L} s_\alpha \cdot \lambda\right) \to \mathrm{U}(\mathfrak{l}_P)_L \otimes_{\mathrm{U}(\mathfrak{b}(\mathfrak{l}_P))_L} \lambda \to W_\lambda \to 0.$$

Proof. See [Hum, Section 9.2]. □

3. Verma modules of finite-dimensional representations

3.1. **An extension of Verma module theory.** Suppose that U is an L-linear finite-dimensional locally analytic representation of the *group* L_P. We remind the reader that this means the L-points of the algebraic group $\mathbf{L}_P$. If $K = \mathbf{Q}_p$ then it is enough to assume that L_P acts continuously by [Eme4, Proposition 3.6.10]. In any case, there is a natural derived action of $\mathrm{U}(\mathfrak{l}_P)_L$ on U, which turns U into a $(\mathfrak{l}_P, L_P)$-module.

We denote by $\mathrm{Ad} : G \to \mathrm{End}_L(\mathrm{U}(\mathfrak{g})_L)$ the adjoint representation of G. Then we endow $M_\mathfrak{p}(U) := \mathrm{U}(\mathfrak{g})_L \otimes_{\mathrm{U}(\mathfrak{p})_L} U$ with an action of L_P by

$$g(X \otimes u) = \mathrm{Ad}(g)X \otimes g(u), \qquad g \in L_P,\ \ X \in \mathrm{U}(\mathfrak{g})_L,\ \ u \in U.$$

The fact that this is well-defined is easily checked and it turns $M_\mathfrak{p}(U)$ into a $(\mathfrak{g}, L_P)$-module. Since $\mathfrak{n}_P$ acts locally finitely on $M_\mathfrak{p}(U)$ this extends naturally to a locally analytic $(\mathfrak{g}, P)$-module structure. To clarify, by a locally analytic $(\mathfrak{g}, P)$-module M we mean that M is an L-linear representation of $\mathfrak{g}$, and an L-linear locally analytic representation of P such that

(1) the derived action of P agrees with the induced action of $\mathfrak{p} \subset \mathfrak{g}$ and
(2) if $g \in P$ and $X \in \mathrm{U}(\mathfrak{g})_L$ and $m \in M$ then $g \cdot (X \cdot m) = (\mathrm{Ad}(g)X)(g \cdot m)$.

The generalized Verma modules $M_\mathfrak{p}(U)$ give examples of elements in the category denoted by $\mathcal{O}^P$ in recent work of Orlik and Strauch [OS2] (see [OS2, Definition 2.3]). The category $\mathcal{O}^P$, by definition, is the full subcategory of locally analytic $(\mathfrak{g}, P)$-modules M such that

($\mathcal{O}^P$-1) $M \in \mathcal{O}^\mathfrak{p}$ as a $\mathrm{U}(\mathfrak{g})_L$-module and
($\mathcal{O}^P$-2) M is an increasing union $M = \bigcup_i M_i$ of finite-dimensional L-linear locally analytic representations of P.

It is explained in [OS2, Example 2.4(ii)] that the P-action is important, i.e. the forgetful map $\mathcal{O}^P \to \mathcal{O}^\mathfrak{p}$ is not surjective in general. However, the issue there is certainly an issue regarding the non-integrability of the action of the torus T.

Denote temporarily by $\mathcal{O}^{\mathfrak{p},T}$ the full subcategory of $(\mathfrak{g}, T)$-modules consisting of locally analytic $(\mathfrak{g}, T)$-modules such that $M \in \mathcal{O}^\mathfrak{p}$ as a $\mathrm{U}(\mathfrak{g})_L$-module and, as above, M is an increasing union of finite-dimensional L-linear locally analytic representations of T.

Proposition 3.1. *The forgetful functor $\mathcal{O}^P \to \mathcal{O}^{\mathfrak{p},T}$ is an equivalence of categories.*

Proof. We need to show that if $M \in \mathcal{O}^{\mathfrak{p},T}$ then there is a canonical action of P on M making it an element of $\mathcal{O}^P$, i.e. if M' is another element of $\mathcal{O}^{\mathfrak{p},T}$ and $M \to M'$ is $(\mathfrak{g},T)$-equivariant then it is automatically P-equivariant for the canonical actions thus defined. Throughout the proof we denote by M and M' two elements of $\mathcal{O}^{\mathfrak{p},T}$.

The action $\mathrm{U}(\mathfrak{n}_P)_L$ on M is locally finite since $M \in \mathcal{O}^{\mathfrak{p}}$ as a $\mathrm{U}(\mathfrak{g})_L$-module. Since N_P is unipotent, we can integrate the action of $\mathfrak{n}_P$ to get a canonical action of N_P on M. Giving M and M' these actions of N_P, it is clear that the map $\mathrm{Hom}_{(\mathfrak{g},N_P)}(M,M') \to \mathrm{Hom}_{\mathfrak{g}}(M,M')$ is a bijection.

Consider now how to extend the T-action on M to an action of the Levi subgroup L_P. Let $B(L_P) = B \cap L_P$ be the Borel subgroup of L_P containing T. It has its own Levi decomposition $B(L_P) = N(L_P)T$ where $N(L_P)$ is the unipotent radical of $B(L_P)$. Since $M \in \mathcal{O}^{\mathfrak{p}}$, the action of $\mathrm{Lie}(N(L_P)) = \mathfrak{n}(\mathfrak{l}_P) \subset \mathfrak{p}$ is locally finite and so M has a canonical action of $N(L_P)$. Since $M \in \mathcal{O}^{\mathfrak{p},T}$, it already comes equipped with a locally analytic action of T and thus M is equipped with a canonical action of the Borel $B(L_P)$ subgroup inside L_P. Moreover, since the action of $N(L_P)$ is obtained via integration and the T-action is given for elements in $\mathcal{O}^{\mathfrak{p},T}$, the canonical map $\mathrm{Hom}_{(\mathfrak{g},B(L_P))}(M,M') \to \mathrm{Hom}_{(\mathfrak{g},T)}(M,M')$ is clearly a bijection also.

Let $N(L_P)^-$ be the unipotent subgroup of L_P opposite to $N(L_P)$. By [Hum, Proposition 9.3(a)], the Lie algebra $\mathfrak{n}(\mathfrak{l}_P)^-$ of $N(L_P)^-$ acts locally finitely of M. Thus M is also equipped with a canonical locally analytic action of the unipotent subgroup $N(L_P)^-$. But now we can also define a canonical action of L_P on M as well. Indeed, it is well-known that L_P is generated by $B(L_P)$ and $N(L_P)^-$ (this follows from the Bruhat decomposition, see [Bor, Corollary 4.14] for example). We can use this to canonically define an action of L_P on M. Explicitly, we write an element x of L_P as a word in $B(L_P)$ and $N(L_P)^-$ and then define the action of x in the obvious way. One must check that this is well-defined, i.e. does not depend on the expression of x as a word, but that is because the action of $N(L_P)^-$ is given by exponentiating the Lie algebra action and M was already a $(\mathfrak{g}, B(L_P))$-module. We deduce as before that giving M and M' the L_P-action thusly, the natural map $\mathrm{Hom}_{(\mathfrak{g},L_P)}(M,M') \to \mathrm{Hom}_{(\mathfrak{g},T)}(M,M')$ is a bijection.

Putting together the previous paragraphs, if M and M' are in $\mathcal{O}^{\mathfrak{p},T}$ and we equip them with the canonical actions of P just described then the natural map

$$\mathrm{Hom}_{(\mathfrak{g},P)}(M,M') \to \mathrm{Hom}_{(\mathfrak{g},T)}(M,M')$$

is a bijection.

It only remains to check that M equipped with the canonical action of P is in $\mathcal{O}^P$, i.e. we need to show that it satisfies the condition ($\mathcal{O}^P$-2). Indeed, by definition of $\mathcal{O}^{\mathfrak{p},T}$, $M = \bigcup M_i$ is an increasing union of finite-dimensional locally analytic T-representations. But M is also an element of $\mathcal{O}^{\mathfrak{p}}$ and so we may assume that each M_i is also stable by the action of $\mathrm{U}(\mathfrak{p})_L$ and $\mathrm{U}(\mathfrak{n}(\mathfrak{l}_P)^-)_L$. But then the action of P on M is determined just by the action of T and the infinitesimal actions of $\mathfrak{p}$ and $\mathfrak{n}(\mathfrak{l}_P)^-$. Thus each M_i is P-stable, which shows ($\mathcal{O}^P$-2). □

Remark 3.2. If we define $\mathcal{O}^{\mathfrak{p},L_P}$ in analogy with $\mathcal{O}^{\mathfrak{p},T}$ then the proof of Proposition 3.1 shows that the equivalence $\mathcal{O}^P \simeq \mathcal{O}^{\mathfrak{p},T}$ factors as two successive equivalences $\mathcal{O}^P \simeq \mathcal{O}^{\mathfrak{p},L_P} \simeq \mathcal{O}^{\mathfrak{p},T}$. Indeed, the first equivalence is the content of the second paragraph of the proof and the remainder of the proof deals with the second equivalence.

Corollary 3.3. *Suppose that* $\mathbf{Q} \supset \mathbf{P}$ *are two choices of standard parabolics. Then the natural forgetful functor* $\mathcal{O}^Q \to \mathcal{O}^P$ *is a fully faithful embedding.*

Proof. The assertion is obvious for the forgetful functor $\mathcal{O}^{\mathfrak{q},T} \to \mathcal{O}^{\mathfrak{p},T}$ and so follows from Proposition 3.1. □

Using Proposition 3.1 we can define canonical extensions of the internal and opposite dualities for Verma modules from Section 2. Recall that we considered two involutions, denoted ι and τ, on $\mathrm{U}(\mathfrak{g})_L$.

Lemma 3.4. *If $t \in T$ and $X \in \mathrm{U}(\mathfrak{g})_L$ then*

(1) $\iota(\mathrm{Ad}(t)X) = \mathrm{Ad}(t)\iota(X)$ *and*
(2) $\tau(\mathrm{Ad}(t)X) = \mathrm{Ad}(t^{-1})\tau(X)$.

Proof. It suffices to consider $X_\alpha \in \mathfrak{g}_\alpha$ for a fixed root α. In that case $\iota(X) \in \mathfrak{g}_\alpha$ and $\tau(X) \in \mathfrak{g}_{-\alpha}$. Both formulas follow easily from this. For example, to see (2) we note that

$$\tau(\mathrm{Ad}(t)X_\alpha) = \alpha(t)\tau(X_\alpha),$$

whereas

$$\mathrm{Ad}(t^{-1})\tau(X_\alpha) = (-\alpha)(t^{-1})\tau(X_\alpha) = \alpha(t)\tau(X_\alpha)$$

(where the root $-\alpha$ is written additively, but it is really the corresponding algebraic character of T, so $(-\alpha)(t^{-1}) = \alpha(t)$). □

Suppose that $M \in \mathcal{O}^P$. We want to define the opposite and internal duals of M with the group structure.

We first define the opposite dual $M^- \in \mathcal{O}^{P^-}$. Consider the dual space M^*. This is equipped with a natural action of T given by

$$(t \cdot f)(m) = f(t^{-1}m).$$

Moreover, Lemma 3.4(1) implies that if we take the $\mathrm{U}(\mathfrak{g})_L$-module $M^{*,\iota}$ then the action of T given here turns $M^{*,\iota}$ into a $(\mathfrak{g}, T)$-module. The action of T also preserves the $\mathfrak{n}_P^-$-finite vectors in $M^{*,\iota}$ and thus we get an induced locally analytic $(\mathfrak{g}, T)$-action on M^-. Since M is in $\mathcal{O}^{\mathfrak{p}}$ it is a weight module for the action of $\mathfrak{t}$, and so is M^-. Thus, the action of T preserves the weight spaces, each of which are finite-dimensional, and so M^- lies in $\mathcal{O}^{\mathfrak{p}^-,T}$. By Proposition 3.1 it defines a canonical element of $\mathcal{O}^{P^-}$ which we also denote by M^-.

Note that the action of T in the previous paragraph does not induce a $(\mathfrak{g}, T)$-module structure on $M^{*,\tau}$. However, by Lemma 3.4(2) the T-module structure on M^* given by

$$(t \cdot f)(m) = f(tm)$$

does induce a $(\mathfrak{g}, T)$-module structure (note that this is a still a left action of T because T is *commutative*). With the action defined, the reasoning in the previous paragraph applies and we see, running through Proposition 3.1, that there is a canonical element $M^\vee \in \mathcal{O}^P$.

Definition 3.5. *If $M \in \mathcal{O}^P$ then its opposite dual is the locally analytic $(\mathfrak{g}, P)$-module $M^- \in \mathcal{O}^{P^-}$ and its internal dual is the locally analytic $(\mathfrak{g}, P)$-module $M^\vee \in \mathcal{O}^P$.*

Proposition 3.6. *The two dualities $M \mapsto M^-$ and $M \mapsto M^\vee$ are exact contravariant functors and there is a commutation relation $(M^\vee)^- \simeq (M^-)^\vee$ in $\mathcal{O}^{P^-}$.*

Proof. The associations are obviously functors. The exactness is a statement about the underlying vector spaces and so follows from Lemma 2.3(3). The commutation relation follows, by Proposition 3.1, from considering first the $\mathrm{U}(\mathfrak{g})_L$-module structures (which is Lemma 2.3(4)) and second the T-module structures (which is clear). □

3.2. **Explicit realizations of some $(\mathfrak{g}, P)$-modules.** In the case of the Verma modules $M_{\mathfrak{p}}(U)$ at the beginning of Section 3.1, one can give other well-known descriptions of their internal and opposite duals. It will be helpful to recall this in order to reference [Eme3]. Throughout this section U will denote an L-linear finite-dimensional representation of L_P.

We denote by U^* the contragradient representation of L_P given by $(g \cdot f)(u) = f(g^{-1}u)$ for all linear functionals $f : U \to L$, $g \in L_P$ and $u \in U$. This is again a locally analytic representation of L_P and as a left $\mathrm{U}(\mathfrak{l}_P)_L$-module this is what we have previously called $U^{*,\iota}$. We can inflate the action of $\mathrm{U}(\mathfrak{l}_P)_L$ on U^* to obtain a left $\mathrm{U}(\mathfrak{p})_L$-module. Now consider the classical adjunction for tensor products

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{U}(\mathfrak{p}^-)_L}(\mathrm{U}(\mathfrak{g})_L, U^*) &\xrightarrow{\simeq} \mathrm{Hom}_L(\mathrm{U}(\mathfrak{g})_L \otimes_{\mathrm{U}(\mathfrak{p}^-)_L} U, L) \\
f &\longmapsto [X \otimes u \mapsto f(\iota X)(u)]\,.
\end{aligned} \tag{6}$$

On the left-hand side of (6), $\mathrm{U}(\mathfrak{g})_L$ is a left $\mathrm{U}(\mathfrak{p}^-)_L$-module via left multiplication and the Hom-space is the abelian group of left $\mathrm{U}(\mathfrak{p}^-)_L$-module morphisms. It is naturally a left $\mathrm{U}(\mathfrak{g})_L$-module via the right regular action on functions; P^- acts on $\mathrm{U}(\mathfrak{p}^-)_L$-equivariant functions $f : \mathrm{U}(\mathfrak{g})_L \to U^*$ via $(pf)(X) = p \cdot f(\mathrm{Ad}(p^{-1})X)$.

The right-hand side of (6), however, is the dual space to a standard Verma module $M_{\mathfrak{p}}(U)$ and we have two separate structures we have considered there, the internal and the opposite dual structures. The presence of ι in the formula suggests which structure on the right-hand side of (6) makes (6) equivariant for the $\mathfrak{g}$-action.

Lemma 3.7. *The tensor-hom adjunction induces an isomorphism of locally analytic $(\mathfrak{g}, P^-)$-modules*

$$\mathrm{Hom}_{\mathrm{U}(\mathfrak{p}^-)_L}(\mathrm{U}(\mathfrak{g})_L, U^*)^{\mathfrak{n}_{P^-}^\infty} \simeq M_{\mathfrak{p}^-}(U)^-.$$

Proof. Observe that the formula in (6) implies that it is an isomorphism of $\mathrm{U}(\mathfrak{g})_L$-modules

$$\mathrm{Hom}_{\mathrm{U}(\mathfrak{p}^-)_L}(\mathrm{U}(\mathfrak{g})_L, U^*) \simeq M_{\mathfrak{p}^-}(U)^{*,\iota}.$$

Moreover, it is evident that this is also equivariant for the action of T on each side. Thus we conclude the lemma by taking finite vectors by $\mathfrak{n}_{P^-}$ and applying Proposition 3.1. □

We now consider the internal dual. Denote by $C^{\mathrm{pol}}(N_P, L)$ the space of L-valued polynomial functions on N_P [Eme2, Section 2.5]. This has a natural action of P where N_P acts via the right regular action and L_P by $(gf)(n) = f(g^{-1}ng)$. We further define $C^{\mathrm{pol}}(N_P, U) := C^{\mathrm{pol}}(N_P, L) \otimes_L U$ with the *diagonal* action of P. Explicitly, if $g \in L_P$ and $n \in N_P$ then

$$(gn \cdot f)(n') = gf(g^{-1}n'ng).$$

This *a fortiori* turns $C^{\mathrm{pol}}(N_P, L)$ into a left $\mathrm{U}(\mathfrak{p})_L$-module. But this action may be upgraded to an action of $\mathrm{U}(\mathfrak{g})_L$. Indeed, for each $f \in C^{\mathrm{pol}}(N_P, L)$ there is a natural L-linear morphism $\mathrm{U}(\mathfrak{n}_P) \to L$ given by $X \mapsto (Xf)(1)$. Taking these together all at once we get an L-linear embedding

$$C^{\mathrm{pol}}(N_P, U) \hookrightarrow \mathrm{Hom}_L(\mathrm{U}(\mathfrak{n}_P)_L, U) \simeq \mathrm{Hom}_{\mathrm{U}(\mathfrak{p}^-)_L}(\mathrm{U}(\mathfrak{g})_L, U), \tag{7}$$

the second isomorphism being *another* tensor-hom adjunction arising from the decomposition $\mathrm{U}(\mathfrak{g})_L \simeq \mathrm{U}(\mathfrak{p}^-)_L \otimes_L \mathrm{U}(\mathfrak{n}_P)_L$. By [Eme3, Lemma 2.5.8], (7) becomes an isomorphism

$$C^{\mathrm{pol}}(N_P, U) \simeq \mathrm{Hom}_{\mathrm{U}(\mathfrak{p}^-)_L}(\mathrm{U}(\mathfrak{g})_L, U)^{\mathfrak{n}_P^\infty} \tag{8}$$

of $\mathrm{U}(\mathfrak{n}_P)_L$-modules after passing to finite vectors for the action of $\mathrm{U}(\mathfrak{n}_P)_L$. The right hand side of (8) has a natural structure of left $\mathrm{U}(\mathfrak{g})_L$-module and we give $C^{\mathrm{pol}}(N_P, U)$ the structure making (8) a $\mathrm{U}(\mathfrak{g})_L$-equivariant isomorphism. The locally analytic $(\mathfrak{g}, P)$-module $C^{\mathrm{pol}}(N_P, U)$ lies in the category $\mathcal{O}^P$ because $C^{\mathrm{pol}}(N_P, U) = \bigcup_d C^{\mathrm{pol}, \leq d}(N_P, U)$, where $C^{\mathrm{pol}, \leq d}(-)$ are the polynomials of degree at most d; each such space is a finite-dimensional P-stable subspace.

On the other hand, the right hand side of the embedding (7) is also the L-linear dual space $M_{\mathfrak{p}^-}(U^*)^*$ to the standard Verma module $M_{\mathfrak{p}^-}(U^*)$. One can easily check that (7) is equivariant for the ι action of $\mathrm{U}(\mathfrak{p})_L$ and P on the target and thus (8) can be written as an isomorphism

$$C^{\mathrm{pol}}(N_P, U) \simeq M_{\mathfrak{p}^-}(U^*)^-$$

in the category $\mathcal{O}^P$. This gives a second reason why $C^{\mathrm{pol}}(N_P, U)$ lies in $\mathcal{O}^P$. We will relate polynomial functions to the internal dual under further restrictions in Section 3.4.

3.3. **Digression on infinitesimally simple representations.** This section uses the notations and conventions of the previous sections. For the reader, we note that we will apply the results of this subsection to the connected split reductive group $\mathbf{L}_P$.

Definition 3.8. *An L-linear finite-dimensional locally analytic representation U of G is called $\mathfrak{g}$-simple if it is irreducible as a module over* $\mathrm{U}(\mathfrak{g})_L$.

Lemma 3.9. *Suppose that U is an L-linear finite-dimensional $\mathfrak{g}$-simple locally analytic representation of G. Then*

(1) *U is irreducible as a G-representation.*
(2) *$\dim_L U^N = 1$ and U is generated as a $L[G]$-module by any non-zero vector in U^N.*
(3) *U is semi-simple as a representation of T.*

Proof. Since U is finite-dimensional, any subspace $U' \subset U$ is closed. Thus if U' were also G-stable, then it would become a subspace stable under the action of $\mathrm{U}(\mathfrak{g})_L$ as well. Since U is assumed to be irreducible as a representation of $\mathrm{U}(\mathfrak{g})_L$, we see that $U' = U$ and this proves (1).

Next we note that the exponential map defines an isomorphism $\exp : \mathfrak{n} \simeq N$ and thus $\dim_L U^N = \dim_L U[\mathfrak{n}]$, where $U[\mathfrak{n}] = \{u \in U \mid \mathfrak{n} \cdot u = 0\}$ is the space of highest weight vectors. But since U is irreducible as a $\mathfrak{g}$-representation, this space is well-known to be one-dimensional. By (1), U is generated as an $L[G]$-module by any non-zero vector $u \in U$ and so *a fortiori* by any non-zero vector in U^N. This proves (2).

If $v \in U^N$ is a non-zero vector, it is a highest weight vector for the derived action. Thus U is generated as a vector space by $\mathrm{U}(\mathfrak{n}^-)_L \cdot v$. Since T acts semi-simply on $\mathrm{U}(\mathfrak{n}^-)_L$ and it acts on v by a character, we conclude (3). □

Suppose that U is an L-linear finite-dimensional $\mathfrak{g}$-simple locally analytic representation of G. By Lemma 3.9 there exists a locally analytic character $\chi : T \to L^\times$ such T acts on U^N through the character χ. Then $d\chi$ is the highest weight of the irreducible finite-dimensional representation U of $\mathrm{U}(\mathfrak{g})_L$. Thus $d\chi \in \Lambda_G^+$.

Proposition 3.10. *The association*

$$\left\{ \begin{array}{c} L\text{-linear finite-dimensional } \mathfrak{g}\text{-simple locally} \\ \text{analytic irreducible representations } U \text{ of } G \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Locally analytic characters} \\ \chi : T \to L^\times \text{ such that } d\chi \in \Lambda_G^+ \end{array} \right\}$$

is a bijection. The inverse is given by

$$\chi \mapsto U_\chi := \operatorname{coker}\left(\bigoplus_{\alpha\in\Delta} M_{\mathfrak{b}}(s_\alpha \cdot \chi) \to M_{\mathfrak{b}}(\chi) \right) \tag{9}$$

Proof. Given U, one can recover χ from the space U^N of N-invariants. Thus the map $U \mapsto \chi$ is injective. It suffices to show that $\chi \mapsto U_\chi$ is well-defined, i.e. U_χ is naturally a locally analytic finite-dimensional representation of G, which is irreducible as a $\mathfrak{g}$-representation and has highest weight χ. But, it is well-known that U_χ, as a representation of $\mathfrak{g}$, is a finite-dimensional irreducible representation of highest weight $d\chi$, i.e. $U_\chi \in \mathcal{O}^{\mathfrak{g}}$. When we equip each factor with their natural locally analytic actions of T, we see that U_χ is canonically an element of the category $\mathcal{O}^{\mathfrak{g},T}$ defined prior to Proposition 3.1. The result now follows from that proposition. $\square$

Let us clarify that the class of $\mathfrak{g}$-simple finite-dimensional locally analytic representations contains many interesting objects.

Proposition 3.11. *If U is an irreducible finite-dimensional locally algebraic representation of G then U is $\mathfrak{g}$-simple.*

Proof. Since U is irreducible, a theorem of Prasad [ST1, Theorem 1, Appendix] shows that $U \simeq U' \otimes \pi$ where U' is an irreducible algebraic representation of G and π is an irreducible smooth representation of G. Each is evidently finite-dimensional since U is. Note that since U' is algebraic, the $\mathfrak{g}$-action on U' is irreducible as well. Moreover, since π is smooth and irreducible, and finite-dimensional, it is one-dimensional. Thus as a $\mathfrak{g}$-module, $U \simeq U'$, meaning that U is irreducible under the derived action of $\mathfrak{g}$. $\square$

Remark 3.12. It is possible that an irreducible finite-dimensional locally analytic representation of G is not irreducible as a $\mathfrak{g}$-module. For example, see [ST1, pg. 120].

3.4. **The internal dual.** We are now ready to relate polynomial functions to the internal dual (compare with Lemma 2.3(5)). We first need a short lemma.

Lemma 3.13. *Let U be an L-linear finite-dimensional locally analytic representation of L_P. Define a finite-dimensional $(\mathfrak{l}_P, T)$-module $\widetilde{U}$ by taking the underlying vector space of $\widetilde{U}$ to be U with the actions given by*

$$X \cdot \widetilde{u} = \widetilde{\iota\tau(X) \cdot u} \qquad (X \in \mathrm{U}(\mathfrak{l}_P)_L)$$
$$t \cdot \widetilde{u} = \widetilde{t^{-1} \cdot u} \qquad (t \in T),$$

(here we write $\widetilde{u} \in \widetilde{U}$ for the element u, but the $\widetilde{(-)}$ is meant to emphasize which action we take). Then the natural map

$$\begin{aligned} \mathrm{Hom}_L(\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p}^-)} \widetilde{U}, L)^\iota &\to \mathrm{Hom}_L(\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p})} U, L)^\tau \\ f &\mapsto \phi_f \end{aligned} \tag{10}$$

given by $\phi_f(X \otimes u) = f(\iota\tau(X) \otimes \widetilde{u})$ is an isomorphism of $(\mathfrak{g}, T)$-modules.

Proof. Let us emphasize that the T-action on the left-hand side of (10) is via $(tf)(z) = f(t^{-1}z)$, whereas on the right-hand side it is via $(tf)(z) = f(tz)$; the $\mathrm{U}(\mathfrak{g})_L$-actions are via the respective involutions. Once this is taken note of, the statement is an elementary check using the actions we have defined. $\square$

Proposition 3.14. *Suppose that U is a finite-dimensional $\mathfrak{l}_P$-simple locally analytic representation of L_P. Then there is a natural isomorphism of locally analytic $(\mathfrak{g}, P)$-modules $M_{\mathfrak{p}^-}(U^*)^- \simeq M_{\mathfrak{p}}(U)^\vee$. In particular, $C^{\mathrm{pol}}(N_P, U) \simeq M_{\mathfrak{p}}(U)^\vee$ as $(\mathfrak{g}, P)$-modules.*

Proof. Since U is $\mathfrak{l}_P$-simple so is U^*. Moreover, if $\widetilde{U}$ is the $(\mathfrak{l}_P, T)$-module as in Lemma 3.13 then it is also irreducible as a $\mathfrak{l}_P$-module. But a comparison of the weights of U^* and $\widetilde{U}$ reveal that we must have $U^* \simeq \widetilde{U}$ as $(\mathfrak{l}_P, T)$-modules; the isomorphism is unique up to a scalar by the irreducibility over $\mathrm{U}(\mathfrak{l}_P)_L$ of either object (compare with the proof of [Bre2, Théorème 4.3]).

Thus by Lemma 3.13, we have isomorphisms of $(\mathfrak{g}, T)$-modules

$$M_{\mathfrak{p}^-}(U^*)^- \simeq \left(\mathrm{Hom}_L(\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p}^-)} \widetilde{U}, L)^\iota\right)^{\mathfrak{n}_P^\infty} \xrightarrow{\simeq} \left(\mathrm{Hom}_L(\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p})} U, L)^\tau\right)^{\mathfrak{n}_P^\infty} = M_{\mathfrak{p}}(U)^\vee.$$

(The first isomorphism, just as $\mathfrak{g}$-modules can also be deduced by Lemma 2.3(5).) By Proposition 3.1, this is automatically a $(\mathfrak{g}, P)$-equivariant isomorphism since the source and the target are in $\mathcal{O}^P$. □

Having realized the internal dual of $M_{\mathfrak{p}}(U)$ concretely, we can now define an explicit $(\mathfrak{g}, P)$-equivariant morphism

$$\alpha_U : M_{\mathfrak{p}}(U) \to M_{\mathfrak{p}}(U)^\vee$$

which generalizes the corresponding map constructed classically in the Verma module theory [Hum, Theorem 3.3(c)] (at least when U is $\mathfrak{l}_P$-simple, which is also the context of the classical theory, compare with [Hum, Observation (1), Section 9.8]).

We continue to assume that U is $\mathfrak{l}_P$-simple. In order to define α_U it is enough, by Proposition 3.14 and (8), to define a $(\mathfrak{g}, P)$-equivariant map

$$M_{\mathfrak{p}}(U) \to \mathrm{Hom}_{\mathrm{U}(\mathfrak{p}^-)_E}(\mathrm{U}(\mathfrak{g})_E, U)^{\mathfrak{n}_P^\infty} = C^{\mathrm{pol}}(N_P, U). \tag{11}$$

As $\mathrm{U}(\mathfrak{g})_L$ is generated over $\mathrm{U}(\mathfrak{p}^-)_L$ by $\mathrm{U}(\mathfrak{n})_L$, we only need to define, for each element of $M_{\mathfrak{p}}(U)$ an L-linear map $\mathrm{U}(\mathfrak{n})_L \to U$. Since $M_{\mathfrak{p}}(U)$ is generated over $\mathrm{U}(\mathfrak{g})_L$ by $1 \otimes u$ for $u \in U$, it is enough to define our maps on these tensors. We do this by putting

$$1 \otimes u \mapsto (\text{constant function } X \mapsto u \text{ on } \mathrm{U}(\mathfrak{n})_L).$$

It is easy to check that this defines a $(\mathfrak{g}, P)$-equivariant morphism, since the action on the target of (11) is the usual left action after evaluating a function.

Proposition 3.15. *Let U be an L-linear finite-dimensional, $\mathfrak{l}_P$-simple representation of L_P.*

(1) *The image of α_U is an irreducible object $L(U)$ in $\mathcal{O}^P$. It is also irreducible as a $\mathrm{U}(\mathfrak{g})_L$-module.*
(2) *Each of $M_{\mathfrak{p}}(U)$ and $M_{\mathfrak{p}}(U)^\vee$ are finite length in $\mathcal{O}^P$ and $L(U)$ appears exactly once as a simple subquotient (resp. submodule) of $M_{\mathfrak{p}}(U)$ (resp. $M_{\mathfrak{p}}(U)^\vee$).*
(3) *The objects $L(U)$ in $\mathcal{O}^P$ are internal self-dual.*

Proof. The image of α_U is in $\mathcal{O}^P$ since $\mathcal{O}^P$ is an abelian category [OS2, Lemma 2.5]. As $\mathrm{U}(\mathfrak{g})_L$-modules, there is a unique, up to scalar, non-zero morphism $M_{\mathfrak{p}}(U) \to M_{\mathfrak{p}}(U)^\vee$ whose image is the unique irreducible $\mathrm{U}(\mathfrak{g})_L$-quotient of $M_{\mathfrak{p}}(U)$. Since α_U is such a map, we see that $L(U)$ is irreducible in $\mathcal{O}^P$. This proves (1).

For part (2), the finite length assertion follows more generally for all of $\mathcal{O}^P$ by [OS2, Lemma 2.7] (it follows just from the assertion for the $\mathrm{U}(\mathfrak{g})_L$-structures; see [Hum, Theorem 1.11]). The single multiplicity of $L(U)$ in $M_{\mathfrak{p}}(U)$ follows from its single multiplicity as a $\mathrm{U}(\mathfrak{g})_L$-module (see Proposition 2.1(4)).

Part (3) follows from part (2) and the exactness of $M \mapsto M^\vee$ given by Proposition 3.6. □

3.5. **Composition factors for Verma modules.** We can push Proposition 3.15 a bit further and determine more precise information about the composition factors in $\mathcal{O}^P$ of $M_{\mathfrak{p}}(U)$ and $M_{\mathfrak{p}}(U)^\vee$ when U is $\mathfrak{l}_P$-simple.

Suppose that $\chi \in \Lambda_P^+$. Then by Proposition 3.10 there is a locally analytic finite-dimensional irreducible representation U_χ of the Levi factor L_P and we can construct the associated Verma module $M_{\mathfrak{p}}(U_\chi) = \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p})} U_\chi$ in $\mathcal{O}^P$. By Proposition 3.15 it has a unique irreducible quotient $L(U_\chi) \in \mathcal{O}^P$.

Recall (see Proposition 2.5) that the representation U_χ may be realized, as a $\mathrm{U}(\mathfrak{l}_P)_L$-module, naturally as a cokernel

$$\bigoplus_{\alpha \in \Delta \setminus I_P} M_{\mathfrak{b}(\mathfrak{l}_P)}(s_\alpha \cdot \chi) \to M_{\mathfrak{b}(\mathfrak{l}_P)}(\chi) \to U_\chi \to 0. \tag{12}$$

The content of Proposition 3.10 (rather, the proof of Proposition 3.1) is that the natural $(\mathfrak{l}_P, B(L_P))$-action everywhere extends to an action of L_P on the quotient in (12). We can inflate the $\mathrm{U}(\mathfrak{l}_P)_L$-action everywhere on (12) to an action of $\mathrm{U}(\mathfrak{p})_L$ and then extend scalars to $\mathrm{U}(\mathfrak{g})_L$. Thus we get an exact sequence

$$\bigoplus_{\alpha\in\Delta\setminus I_P} M_{\mathfrak{b}}(s_\alpha\cdot\chi)\to M_{\mathfrak{b}}(\chi)\to M_{\mathfrak{p}}(U_\chi)\to 0 \tag{13}$$

in the category $\mathcal{O}^B$. By Corollary 3.3, the $(\mathfrak{g},P)$-module structure on the quotient in (13) is uniquely determined by this sequence.

Lemma 3.16. *If $\chi\in\Lambda_P^+$ then $L(\chi)\simeq L(U_\chi)$ in $\mathcal{O}^B$. In particular, $L(\chi)\in\mathcal{O}^P$.*

Proof. The $(\mathfrak{g},P)$-module $M_{\mathfrak{p}}(U_\chi)$ is a quotient in $\mathcal{O}^B$ of $M_{\mathfrak{b}}(\chi)$. Thus $L(U_\chi)$ is a quotient of $M_{\mathfrak{b}}(\chi)$ in $\mathcal{O}^B$. But $L(\chi)$ is the unique such quotient by Proposition 3.15 and thus $L(\chi)\simeq L(U_\chi)$ as $(\mathfrak{g},B)$-modules. The final assertion then follows from Corollary 3.3. □

Theorem 3.17. *Suppose that $\chi\in\Lambda_P^+$. Then:*

1. *Every irreducible constituent of $M_{\mathfrak{p}}(U_\chi)$ in $\mathcal{O}^P$ is of the form $L(\chi')$ for some $\chi'\in\Lambda_P^+$ which is strongly linked to χ.*
2. *The list of irreducible constituents of $M_{\mathfrak{p}}(U_\chi)$ and $M_{\mathfrak{p}}(U_\chi)^\vee$ are the same, with multiplicity.*

The definition of *strongly linked* is recalled in the appendix (see Definition A.2). It is an analog of the notation of strongly linked weights coming from the theory of Verma modules [Hum].

Proof. Part (2) follows from part (1) and Proposition 3.15(3). It remains to show part (1). We remark that we have shown part (1) in the case where $\mathbf{P}=\mathbf{B}$ in [BC, Lemma 4.2]. To keep this paper self-contained, we reproduce the argument in the appendix (see Proposition A.4).

Suppose that Λ is an irreducible constituent of $M_{\mathfrak{p}}(U_\chi)$ as an object of $\mathcal{O}^P$. Then it contains a simple $\mathcal{O}^B$-object $L\subset\Lambda$. Since $M_{\mathfrak{p}}(U_\chi)$ is a $(\mathfrak{g},B)$-quotient of $M_{\mathfrak{b}}(\chi)$ it follows that L is a constituent of of $M_{\mathfrak{b}}(\chi)$. Thus by case $B=P$ we know that $L=L(\chi')$ for some χ' strongly linked to χ. We claim that $\chi'\in\Lambda_P^+$.

To see that $\chi'\in\Lambda_P^+$, first note that this depends only on the derivative $d\chi'$, i.e. it is only a question of the underlying $\mathrm{U}(\mathfrak{g})_L$-modules. Second note that $L(\chi')$ is irreducible as a $\mathrm{U}(\mathfrak{g})_L$-module by Proposition 3.15(1) and thus a $\mathrm{U}(\mathfrak{g})_L$-module constituent of $M_{\mathfrak{p}}(U_\chi)$. Finally, this implies $\chi'\in\Lambda_P^+$ by [Hum, Proposition 9.3(e)].

Finally, since $\chi'\in\Lambda_P^+$ we know that $L(\chi')\in\mathcal{O}^P$ by Lemma 3.16. Since $L(\chi')$ is irreducible and Λ is irreducible in $\mathcal{O}^P$ we conclude that $\Lambda=L(\chi')$ and we are done. □

Remark 3.18. In the case where $\mathbf{P}=\mathbf{B}$, Theorem 3.17 can be made more precise by specifying that $L(\chi')$ appears in $M_{\mathfrak{b}}(\chi)$ as an element of $\mathcal{O}^B$ for all χ' strongly linked to χ (see Proposition A.4). The obstruction to a similar amount of precision for $M_{\mathfrak{p}}(U_\chi)$ lies entirely within difficulties in the BGG category $\mathcal{O}^{\mathfrak{p}}$.

Remark 3.19. It would be interesting to understand the analog of Theorem 3.17 for the generalized Verma modules $M_{\mathfrak{p}}(U)$ when U is not $\mathfrak{l}_P$-simple.

4. The adjunction formula

Our adjunction formula relates certain locally analytic principle series with the Emerton–Jacquet functor of a locally analytic representation of the group G.

Definition 4.1. *Let V be an L-linear locally analytic representation of G.*

(1) *V is called very strongly admissible if there exists a continuous L-linear and G-equivariant injection $V \hookrightarrow B$ where B is a continuous admissible representation of G on an L-Banach space.*
(2) *V is called $\mathfrak{p}$-acyclic if $\mathrm{Ext}^1_{\mathfrak{p}}(-, V) = (0)$, i.e. if V is injective as a module over $\mathrm{U}(\mathfrak{p})_L$.*

To a very strongly admissible locally analytic representation V of G (but also more generally), Emerton has associated a locally analytic representation $J_P(V)$ of L_P, which we call the Emerton–Jacquet module of V [Eme2, Eme3]. We do not recall the precise definition here, only it's relevant properties. By [Eme2, Theorem 0.3], it is characterized as being right adjoint to the functor $U \mapsto \mathcal{C}_c^{\mathrm{sm}}(N_P, U)$ on L-linear locally analytic representations of L_P (on compact type spaces). If $N_P^0 \subset N_P$ is a fixed compact open subgroup then, by definition, the space $J_P(V)$ is closely related to the space of invariants $V^{N_P^0}$ [Eme2, Definition 3.4.5] and the action by L_P should be thought of as extending the action of the monoid L_P^+ on $V^{N_P^0}$ as described at the end of the introduction (see Section 1.6 and below).

When V is very strongly admissible, the locally analytic action of the center $Z(L_P)$ on $J_P(V)$ extends to an action of the space $\mathcal{O}(\widehat{Z(L_P)})$ of rigid analytic functions on the p-adic rigid space $\widehat{Z(L_P)}$ parameterizing locally analytic characters of $Z(L_P)$ (this is part of the content of [Eme1, Theorem 0.5]).

If $\eta \in \widehat{Z(L_P)}$ is a character then the set of functions vanishing at η defines a maximal ideal $\mathfrak{m}_\eta \subset \mathcal{O}(\widehat{Z(L_P)})$. We write $J_P(V)^\eta$ for the closed L_P-stable subspace which is annihilated by $\mathfrak{m}_\eta$. Equivalently, $J_P(V)^\eta$ is the closed subspace of $J_P(V)$ on which $Z(L_P)$ acts through the character η. One may equivalently describe $J_P(V)^\eta$ as the space $V^{N_P^0, Z(L_P)^+=\eta}$ where $Z(L_P)^+$ is the monoid $Z(L_P)^+ = L_P^+ \cap Z(L_P)$ and the action of $t \in Z(L_P)^+$ is through the action π_t described in the notations section (Section 1.6).

We are now ready to give the key definition.

Definition 4.2. *Suppose that*

- *$U = U_\chi$ is an L-linear finite-dimensional $\mathrm{U}(\mathfrak{l}_P)_L$-simple representation of L_P,*
- *π an L-linear smooth representation of finite length admitting a central character ω_π and*
- *V is an L-linear very strongly admissible locally analytic representation of G.*

We say that the pair (U, π) is non-critical with respect to V if

(1) $\mathrm{Hom}_{L_P}(U \otimes \pi, J_P(V)) \neq (0)$ *and*
(2) *if $\chi' \neq \chi$ is a locally analytic character of T, such that $L(\chi')$ is a Jordan–Holder factor of $M_{\mathfrak{p}}(U_\chi)$, then $J_P(V)^{Z(L_P)=\chi'\omega_\pi} = (0)$.*

Remark 4.3. The first condition implies that (U, π) being non-critical depends on more than ω_π.

Remark 4.4. The qualification over characters of the center in the second condition is inspired by the Borel case and practicality. See Section 1.5. The ambiguity in which characters χ' need to checked is already subtle on the level of Verma modules.

Remark 4.5. Since V is assumed to be very strongly admissible, [Eme2, Lemma 4.4.2] provides a sufficient condition[1] for (U, π) to be non-critical, independent of V. Indeed, it is sufficient that the character $\chi\omega_\pi$ be of *non-critical slope* in the sense of [Eme2, Definition 4.4.3] (at least when χ is assumed to be locally algebraic).

[1]There is a typo in [Eme2, Lemma 4.4.2]. The slope (in the sense of [Eme2]) of the modulus character is -2ρ, not $-\rho$ (here ρ is the half-sum of positive roots, what we are denoting ρ_0). Thus, ρ should be replaced by 2ρ in the statement of [Eme2, Lemma 4.4.2]. Despite this, [Eme2, Definition 4.4.3] should remain unchanged.

Example 4.6. Suppose that $\mathbf{G} = \mathrm{GL}_{n/\mathbf{Q}_p}$, $\mathbf{P} = \mathbf{B}$ is the upper triangular Borel, ψ_k is the algebraic character of dominant weight $k_1 \geq k_2 \geq \cdots \geq k_n$ and π is a smooth character. Put $\chi = \psi_k\pi$ and $\tilde{\chi} = \chi\delta_B^{-1}$. Let x_i be the diagonal element of T given by $\mathrm{diag}(p, \dots, p, 1, \dots, 1)$, where there are i different p's. Then it is easy to see that χ is of non-critical slope if and only if $v_p(\widetilde{\chi}(x_i)) < k_i - k_{i+1} + 1$ for all $1 \leq i \leq n-1$.

Condition (2) in Definition 4.2 implies a more natural looking condition.

Lemma 4.7. *Suppose that U, π and V are a triple as in Definition 4.2 and that (U, π) is non-critical with respect to V. If $\chi' \neq \chi$ is a locally analytic character and $L(\chi')$ is a Jordan–Holder factor of $M_{\mathfrak{p}}(U_\chi)$ then* $\mathrm{Hom}_{L_P}(U_{\chi'} \otimes \pi, J_P(V)) = (0)$.

Proof. This is immediate from the condition (2). □

In applications, we want to relate the L_P-isotypic spaces as in Lemma 4.7 to issues about Verma modules. For that, we have the following proposition, which follows from the description of the Emerton–Jacquet module as a right adjoint functor.

Proposition 4.8. *Suppose that V is an L-linear very strongly admissible representation of G and $U' = U \otimes \pi$ is the tensor product of an L-linear finite-dimensional locally analytic representation U of L_P and an L-linear smooth representation π of L_P. Then there exists a canonical isomorphism*

$$\mathrm{Hom}_{(\mathfrak{g},P)}\left(M_{\mathfrak{p}}(U) \otimes_L \mathcal{C}_c^{\mathrm{sm}}(N_P, \pi(\delta_P^{-1})), V\right) \xrightarrow{\simeq} \mathrm{Hom}_{L_P}\left(U', J_P(V)\right). \tag{14}$$

Proof. First note that $M_{\mathfrak{p}}(U) \otimes_L \mathcal{C}_c^{\mathrm{sm}}(N_P, \pi(\delta_P^{-1})) \simeq \mathrm{U}(\mathfrak{g})_L \otimes_{\mathrm{U}(\mathfrak{p})_L} \mathcal{C}_c^{\mathrm{sm}}(N_P, U'(\delta_P^{-1}))$. Then the statement follows from [Eme2, Theorem 3.5.6] (and also [Eme3, (0.17)]). □

We now state our main theorem.

Theorem 4.9. *Suppose that*

- *V is an L-linear very strongly admissible, $\mathfrak{p}$-acyclic, representation of G,*
- *U is an L-linear finite-dimensional $\mathfrak{l}_P$-simple locally analytic representation of L_P and*
- *π is a finite length smooth representation of L_P admitting a central character.*

If the pair (U, π) is non-critical with respect to V then there exists a canonical isomorphism

$$\mathrm{Hom}_G\left(\mathrm{Ind}_{P^-}^G(U \otimes \pi(\delta_P^{-1}))^{\mathrm{an}}, V\right) \simeq \mathrm{Hom}_{L_P}\left(U \otimes \pi, J_P(V)\right).$$

A version of this theorem in the case that $\mathbf{P} = \mathbf{B}$ and $\mathbf{G} = \mathrm{GL}_{3/\mathbf{Q}_p}$ (though that assumption was not crucially used) was proven in [BC, Theorem 4.1]. The proof we give here is similar, though there are several new details because of the generality we work in.

Throughout the rest of this section we fix U, π and V as in the statement of Theorem 4.9. We also set $U' = U \otimes \pi(\delta_P^{-1})$, which is a locally analytic representation of L_P.

Let $\mathcal{C}_c^{\mathrm{lp}}(N_P, L)$ be the space of compactly supported L-valued locally polynomial functions on N_P. If W is any locally analytic representation of L_P then we denote by $\mathcal{C}_c^{\mathrm{lp}}(N_P, W)$ the W-valued locally polynomial functions. Because of the natural open immersion $N_P \hookrightarrow G/P^-$, we can regard $\mathcal{C}_c^{\mathrm{lp}}(N_P, U')$ as a $(\mathfrak{g}, P)$-stable subspace of $\mathrm{Ind}_{P^-}^G(U')^{\mathrm{an}}$. The inclusion of $\mathcal{C}_c^{\mathrm{sm}}(N_P, U')$ in $\mathcal{C}_c^{\mathrm{lp}}(N_P, U')$ thus induces a $(\mathfrak{g}, P)$-equivariant map

$$\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p}^-)} \mathcal{C}_c^{\mathrm{sm}}(N_P, U') \to \mathcal{C}_c^{\mathrm{lp}}(N_P, U'). \tag{15}$$

We consider the following diagram (which is commutative, as we will explain):

$$
\begin{array}{ccc}
\mathrm{Hom}_G(\mathrm{Ind}_{P^-}^G(U')^{\mathrm{an}}, V) & \xrightarrow{(1)} & \mathrm{Hom}_{L_P}(U \otimes \pi, J_P(V)) \\
{\scriptstyle (a)}\downarrow\simeq & & \simeq\downarrow{\scriptstyle (b)} \\
\mathrm{Hom}_{(\mathfrak{g},P)}(\mathcal{C}_c^{\mathrm{lp}}(N_P, U'), V) & \xrightarrow{(2)} & \mathrm{Hom}_{(\mathfrak{g},P)}(M_{\mathfrak{p}}(U) \otimes \mathcal{C}_c^{\mathrm{sm}}(N_P, \pi(\delta_P^{-1})), V) \\
{\scriptstyle (c)}\downarrow\simeq & & \| \\
\mathrm{Hom}_{(\mathfrak{g},P)}(M_{\mathfrak{p}}(U)^\vee \otimes \mathcal{C}_c^{\mathrm{sm}}(N_P, \pi(\delta_P^{-1})), V) & \xrightarrow{(3)} & \mathrm{Hom}_{(\mathfrak{g},P)}(M_{\mathfrak{p}}(U) \otimes \mathcal{C}_c^{\mathrm{sm}}(N_P, \pi(\delta_P^{-1})), V) \\
{\scriptstyle (3a)}\downarrow & \nearrow{\scriptstyle (3b)} & \\
\mathrm{Hom}_{(\mathfrak{g},P)}(L(U) \otimes \mathcal{C}_c^{\mathrm{sm}}(N_P, \pi(\delta_P^{-1})), V) & &
\end{array}
$$

The statement of Theorem 4.9 is that the map (1) is an isomorphism. Let us now explain all the identifications and maps.

Let $\mathrm{Ind}_{P^-}^G(U')(N_P)$ denote the subspace of $\mathrm{Ind}_{P^-}^G(U')^{\mathrm{an}}$ consisting of functions supported on N_P. By [Eme3, Theorem 4.1.5] and [Eme3, Theorem 4.2.18] we have a sequence of canonical isomorphisms

$$
\begin{aligned}
\mathrm{Hom}_G(\mathrm{Ind}_{P^-}^G(U')^{\mathrm{an}}, V) &\simeq \mathrm{Hom}_{(\mathfrak{g},P)}(\mathrm{Ind}_{P^-}^G(U')(N_P), V) \\
&\simeq \mathrm{Hom}_{(\mathfrak{g},P)}(\mathcal{C}_c^{\mathrm{lp}}(N_P, U'), V).
\end{aligned}
$$

This gives the identification (a). Since $\mathcal{C}_c^{\mathrm{lp}}(N_P, U') \simeq \mathcal{C}^{\mathrm{pol}}(N_P, U) \otimes_L \mathcal{C}_c^{\mathrm{sm}}(N_P, \pi(\delta_P^{-1}))$, the identification (c) follows from Proposition 3.14. The identification (b) is Emerton's adjoint definition of the Emerton–Jacquet functor (see Proposition 4.8).

The map (2) is induced by the natural map of $(\mathfrak{g}, P)$-modules

$$M_{\mathfrak{p}}(U) = \mathrm{U}(\mathfrak{g})_L \otimes_{\mathrm{U}(\mathfrak{p})_L} U \to \mathcal{C}^{\mathrm{pol}}(N_P, U)$$

given by differentiation. The map (3) is induced from the map

$$\alpha_U : M_{\mathfrak{p}}(U) \to M_{\mathfrak{p}}(U)^\vee$$

defined in Section 3.4. The commutation between (2) and (3) is realized by Proposition 3.14. By Proposition 3.15, α_U factors as $M_{\mathfrak{p}}(U) \twoheadrightarrow L(U) \hookrightarrow M_{\mathfrak{p}}(U)^\vee$, which gives the maps (3a) and (3b). The map (3b) is always injective since $L(U)$ is a quotient of $M(U)$.

We will now prove Theorem 4.9 by showing that the map (3) is an isomorphism. In turn we will show separately that the maps (3a) and (3b) are isomorphisms.

Proposition 4.10. *The map (3b) is an isomorphism and (3a) is injective.*

Proof. Let $M' = \ker(M_{\mathfrak{p}}(U) \to L(U))$. M' lies in the category $\mathcal{O}^P$. Since $L(U)$ appears exactly once as a constituent of $M_{\mathfrak{p}}(U)$ we get, by repeated applications of Definition 4.2, Lemma 4.7 and Proposition 4.8, that $\mathrm{Hom}_{(\mathfrak{g},P)}(M' \otimes_L \mathcal{C}_c^{\mathrm{sm}}(N_P, \pi(\delta_P^{-1})), V) = (0)$. Since $\mathrm{Hom}_{(\mathfrak{g},P)}(-, V)$ is left-exact we see that (3b) is surjective. We already remarked that (3b) is injective, and thus (3b) is an isomorphism.

To consider (3a) we take internal duals and look at

$$0 \to L(U) \to M_{\mathfrak{p}}(U)^\vee \to (M')^\vee \to 0.$$

The irreducible constituents of $(M')^\vee$ are the constituents appearing in $M_{\mathfrak{p}}(U)^\vee$, except for $L(U)$. These are the *same* as the constituents appearing in $M_{\mathfrak{p}}(U)$ except for $L(U)$ by Proposition 3.15(3). In particular, we apply the same reasoning from the previous paragraph to see that (3a) is injective. □

To finish the proof of the Theorem 4.9 we need to prove that (3a) is surjective.

Lemma 4.11. *Let V and M be $(\mathfrak{g}, P)$-modules which are locally analytic representations of P on compact type spaces and π a smooth representation of L_P. Then we have natural isomorphisms*

$$\begin{aligned}\mathrm{Hom}_{(\mathfrak{g},P)}(M \otimes \mathcal{C}_c^{\mathrm{sm}}(N_P, \pi(\delta_P^{-1})), V) &\simeq \mathrm{Hom}_P(\mathcal{C}_c^{\mathrm{sm}}(N_P, \pi(\delta_P^{-1})), \mathrm{Hom}_{\mathfrak{g}}(M, V)) \\ &\simeq \mathrm{Hom}_{L_P}(\pi, J_P(\mathrm{Hom}_{\mathfrak{g}}(M, V))).\end{aligned}$$

Proof. The first map is the canonical map from the tensor-hom adjunction. The second map is Emerton's adjunction formula for the Emerton–Jacquet functor [Eme2, Theorem 3.5.6] (which is easier here because both spaces are smooth representations of L_P). □

Recall that $U = U_\chi$ has highest weight χ.

Lemma 4.12. *If $\chi' \neq \chi$ and $L(\chi')$ is a constituent of $M_{\mathfrak{p}}(U_\chi)$ then*

$$J_P(\mathrm{Hom}_{\mathfrak{g}}(L(\chi'), V))_{\mathfrak{m}_{\omega_\pi}} = (0).$$

Proof. Let $X = \mathrm{Hom}_{\mathfrak{g}}(L(\chi'), V)$. Since localization commutes with dualizing it is enough to show that the same is true when we replace $J_P(X)$ by its continuous dual $J_P(X)'$. The space $J_P(X)'$ is the global sections of a coherent sheaf on $\widehat{Z(L_P)}$ and hence the localization at $\mathfrak{m}_{\omega_\pi}$ vanishes if and only if $J_P(X)'/\mathfrak{m}_{\omega_\pi} = (0)$. But there is a natural duality between $J_P(X)'/\mathfrak{m}_{\omega_\pi}$ and $J_P(X)^{Z(L_P)=\omega_\pi}$, and so it suffices to show that $J_P(X)^{Z(L_P)=\omega_\pi}$ vanishes.

Let $e_{\chi'}^+$ be the highest weight vector in $L(\chi')$, the center $Z(L_P)$ acts on $e_{\chi'}^+$ via χ'. Then we have a canonical embedding

$$\begin{aligned}J_P(\mathrm{Hom}_{\mathfrak{g}}(L(\chi'), V))^{Z(L_P)=\omega_\pi} &\hookrightarrow J_P(V)^{Z(L_P)=\omega_\pi\chi'} \\ f &\mapsto f(e_{\chi'}^+).\end{aligned}$$

But by the definition of non-critical, the target vanishes and so we are done. □

Lemma 4.13. *If $\chi' \neq \chi$ and $L(\chi')$ is a constituent of $M_{\mathfrak{p}}(U_\chi)$ then*

$$J_P(\mathrm{Ext}^1_{\mathfrak{g}}(L(\chi'), V))_{\mathfrak{m}_{\omega_\pi}} = (0).$$

Proof. The first thing to note is that $\mathrm{Ext}^1_{\mathfrak{p}}(U_\chi, V) = (0)$ since V is $\mathfrak{p}$-acyclic, and thus by Shapiro's lemma we have $\mathrm{Ext}^1_{\mathfrak{g}}(M_{\mathfrak{p}}(U_\chi), V) = (0)$.

First suppose that M is a proper submodule of $M_{\mathfrak{p}}(U_\chi)$ in the category $\mathcal{O}^P$. Then we have an exact sequence

$$0 \to \mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(U_\chi)/M, V) \to \mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(U_\chi, V)) \to \mathrm{Hom}_{\mathfrak{g}}(M, V) \to \mathrm{Ext}^1_{\mathfrak{g}}(M_{\mathfrak{p}}(U_\chi)/M, V) \to 0.$$

Note that since $\mathfrak{g}$ acts trivially everywhere, this is an exact sequence of *smooth* representations of P and thus remains exact after applying the Emerton–Jacquet functor (which is just the usual Jacquet module on smooth representations). After localizing at $\mathfrak{m}_{\omega_\pi}$ we see that we have a surjection

$$J_P(\mathrm{Hom}_{\mathfrak{g}}(M, V))_{\mathfrak{m}_{\omega_\pi}} \to J_P(\mathrm{Ext}^1_{\mathfrak{g}}(M_{\mathfrak{p}}(U_\chi)/M, V))_{\mathfrak{m}_{\omega_\pi}} \to 0.$$

But Lemma 4.12 implies, by dévissage, that $J_P(\mathrm{Hom}_{\mathfrak{g}}(M, V))_{\mathfrak{m}_{\omega_\pi}} = (0)$. We deduce that

$$J_P(\mathrm{Ext}^1_{\mathfrak{g}}(M_{\mathfrak{p}}(U_\chi)/M, V))_{\mathfrak{m}_{\omega_\pi}} = (0). \tag{16}$$

Now suppose $\chi' \neq \chi$ and $L(\chi')$ is an irreducible constituent of $M_{\mathfrak{p}}(U_\chi)$. Since $\chi \neq \chi'$, there exists a *proper* submodule M of $M_{\mathfrak{p}}(U_\chi)$ such that $L(\chi')$ is a quotient of $M_{\mathfrak{p}}(U_\chi)/M$. If $N = \ker(M_{\mathfrak{p}}(U_\chi)/M \twoheadrightarrow L(\chi'))$ then, we deduce from (16) (and using the same exactness argument as before) that we have an exact sequence

$$(17)\quad 0 \to J_P(\mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(U_\chi)/M, V))_{\mathfrak{m}_{\omega_\pi}} \to J_P(\mathrm{Hom}_{\mathfrak{g}}(N, V))_{\mathfrak{m}_{\omega_\pi}} \to J_P(\mathrm{Ext}^1_{\mathfrak{g}}(L(\chi'), V))_{\mathfrak{m}_{\omega_\pi}} \to 0.$$

We claim that the first non-zero map in (17) is an isomorphism, which would complete the proof.

Let's prove the claim. Since M is proper we know that the natural map

$$(18)\qquad J_P(\mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(U_\chi)/M, V))_{\mathfrak{m}_{\omega_\pi}} \to J_P(\mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(U_\chi), V))_{\mathfrak{m}_{\omega_\pi}}$$

is an isomorphism by the argument in the second paragraph of this proof.

Since $L(\chi)$ does not appear as a composition factors of M, and $\chi \neq \chi'$, $L(\chi)$ does appear as a composition factor of N, and in particular is a quotient of N. By dévissage from Lemma 4.12 we deduce that the natural map

$$(19)\qquad J_P(\mathrm{Hom}_{\mathfrak{g}}(L(\chi), V))_{\mathfrak{m}_{\omega_\pi}} \to J_P(\mathrm{Hom}_{\mathfrak{g}}(N, V))_{\mathfrak{m}_{\omega_\pi}}$$

is an isomorphism also.

Finally, we know that the natural map $J_P(\mathrm{Hom}_{\mathfrak{g}}(L(\chi), V))_{\mathfrak{m}_{\omega_\pi}} \to J_P(\mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(\chi), V))_{\mathfrak{m}_{\omega_\pi}}$ is an isomorphism as well, using Lemma 4.12 again to perform the dévissage. Thus combining (18) and (19), we deduce that the two spaces at the beginning of (17) have the same dimension. We conclude then that the first map of (17) is an isomorphism. □

Remark 4.14. If V is $\mathfrak{g}$-acyclic then Lemma 4.13 becomes a tautology. However, there are interesting situations where V is only known to be $\mathfrak{p}$-acyclic, for example the p-adically completed (compactly supported) cohomology of a tower of modular curves [BE, Section 5.1].

We now finish the proof of Theorem 4.9.

Proof of Theorem 4.9. Recall that following Proposition 4.10 it remains to show that the map labeled (3a) in the diagram on page 17 is surjective. Consider the short exact sequence of in $\mathcal{O}^P$

$$(20)\qquad 0 \to L(U) \to M_{\mathfrak{p}}(U)^\vee \to (M')^\vee \to 0.$$

Consider the long exact sequence

$$0 \to \mathrm{Hom}_{\mathfrak{g}}((M')^\vee, V) \to \mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(U)^\vee, V) \to \mathrm{Hom}_{\mathfrak{g}}(L(U), V) \to \mathrm{Ext}^1_{\mathfrak{g}}((M')^\vee, V) \to \cdots$$

of *smooth* representations of P. Since the Emerton–Jacquet module is exact on the smooth category, and localization is exact, we deduce an exact sequence

$$(21)\quad 0 \to J_P(\mathrm{Hom}_{\mathfrak{g}}((M')^\vee, V))_{\mathfrak{m}_{\omega_\pi}} \to J_P(\mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(U)^\vee, V))_{\mathfrak{m}_{\omega_\pi}} \to J_P(\mathrm{Hom}_{\mathfrak{g}}(L(U), V))_{\mathfrak{m}_{\omega_\pi}} \to J_P(\mathrm{Ext}^1_{\mathfrak{g}}((M')^\vee, V))_{\mathfrak{m}_{\omega_\pi}} \to \cdots.$$

A dévissage from Lemma 4.13 implies that

$$(22)\qquad J_P(\mathrm{Ext}^1_{\mathfrak{g}}((M')^\vee, V))_{\mathfrak{m}_{\omega_\pi}} = (0)$$

because the irreducible constituents of $(M')^\vee$ are of the form $L(\chi')$ with $\chi' \neq \chi$ by Theorem 3.17. On the other hand, the same dévissage using Lemma 4.12 implies that

$$J_P(\mathrm{Hom}_{\mathfrak{g}}((M')^\vee, V))_{\mathfrak{m}_{\omega_\pi}} = (0).$$

Thus we deduce from (21) that the natural map

$$J_P(\mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(U)^\vee, V))_{\mathfrak{m}_{\omega_\pi}} \to J_P(\mathrm{Hom}_{\mathfrak{g}}(L(U), V))_{\mathfrak{m}_{\omega_\pi}}$$

is an isomorphism. In particular, after taking $\mathfrak{m}_{\omega_\pi}$-torsion we also get an equality

$$\mathrm{Hom}_{Z(L_P)}(\omega_\pi, J_P(\mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(U)^\vee, V))) = \mathrm{Hom}_{Z(L_P)}(\omega_\pi, J_P(\mathrm{Hom}_{\mathfrak{g}}(L(U), V))).$$

But this easily implies that that we get a natural equality

$$\mathrm{Hom}_{L_P}(\pi, J_P(\mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(U)^\vee, V))) = \mathrm{Hom}_{L_P}(\pi, J_P(\mathrm{Hom}_{\mathfrak{g}}(L(U), V)))$$

(see Lemma 4.15 below—take $\Gamma = L_P$, $C = Z(L_P)$ and the obvious choices for V, V' and V''). By Lemma 4.12, this is the equality we seek for (3a), and so we are done. □

We left a lemma at the end of the argument. We leave the details of its proof to the reader.

Lemma 4.15. *Suppose that Γ is a group and $C \subset \Gamma$ is a central subgroup. If V, V', V'' are Γ-modules and $f : V' \to V''$ is a Γ-equivariant map such that the induced morphism*

$$\mathrm{Hom}_C(V, V') \to \mathrm{Hom}_C(V, V'')$$

is an isomorphism then the natural map

$$\mathrm{Hom}_\Gamma(V, V') \to \mathrm{Hom}_\Gamma(V, V'')$$

is an isomorphism.

Proof. If W and U are any two Γ-modules then the group Γ acts on $\mathrm{Hom}_C(W, U)$ by the formula $(\gamma f)(w) = \gamma f(\gamma^{-1}w)$ for all $\gamma \in \Gamma$, $w \in W$ and $f \in \mathrm{Hom}_C(W, U)$. This is well-defined, and functorial in Γ-modules, *because* C is a central subgroup in Γ. Thus the lemma follows just from the observation that $\mathrm{Hom}_\Gamma(W, U)$ is the fixed points $\mathrm{Hom}_C(W, U)^\Gamma$ for this action. □

5. Restriction to the socle

The goal of this section to improve Theorem 4.9. Namely, we study the restriction of morphisms $\mathrm{Ind}_{P^-}^G(U \otimes \pi(\delta_P^{-1})) \to V$ which arise via adjunction in the non-critical case to the *locally analytic socle*. Unlike the rest of this article, we restrict here to the case of finite-dimensional irreducible *algebraic* representations U of L_P, i.e. the canonical action of L_P on a finite-dimensional irreducible algebraic representation of the underlying reductive group $\mathbf{L}_P$. This necessary to make use of results of Breuil in [Bre2], extending recent progress of Orlik–Strauch [OS1, OS2] on studying the locally analytic principal series.

5.1. **The Orlik–Strauch representations.** Suppose that $M \in \mathcal{O}^P$ and π is a smooth admissible representation of L_P. In [OS2] (and [OS1]) Orlik and Strauch define a locally analytic representation $\mathcal{F}_P^G(M, \pi)$ of the group G. The association $(M, \pi) \mapsto \mathcal{F}_P^G(M, \pi)$ is a functor, contravariant in M and covariant in π. It is exact in both arguments (see [OS2, Section 3] for these results). When U is a finite-dimensional locally analytic representation of L_P we can consider the generalized Verma module $M_{\mathfrak{p}}(U) \in \mathcal{O}^P$. The definition of the Orlik–Strauch representations immediately gives

$$\mathcal{F}_P^G(M_{\mathfrak{p}}(U), \pi) = \mathrm{Ind}_P^G(U^* \otimes \pi)^{\mathrm{an}},$$

so these representations naturally include locally analytic principal series (note that the representation U^* on the right is the *dual* of the representation U on the left).

Recall that if $M \in \mathcal{O}^P$ then we say that $\mathfrak{p}$ is *maximal* for M if $M \notin \mathcal{O}^{\mathfrak{q}}$ for all $\mathfrak{q} \supsetneq \mathfrak{p}$. Since $\mathcal{F}_P^G(M, \pi)$ is exact in both arguments there are obvious necessary conditions for the irreducibility. Orlik and Strauch established sufficient conditions.

Theorem 5.1 ([OS2, Theorem 1.1])**.** *If $M \in \mathcal{O}^P$, $\mathfrak{p}$ is maximal for M and π is an irreducible smooth representation of L_P then $\mathcal{F}_P^G(M, \pi)$ is topologically irreducible.*

Remark 5.2. It is possible to deduce the irreducibility in some cases without assuming that P is the maximal parabolic. This is done by using a relation (the "PQ"-formula) between $\mathcal{F}_Q^G(-,-)$ and $\mathcal{F}_P^G(-,-)$ for a containment $P \subset Q$ of a parabolics. See the proof Theorem 5.4.

5.2. **An adjunction formula with the socle in the algebraic case.** Prior to the work [OS2], results mentioned in the previous section (especially Theorem 5.1) were established in the algebraic case (see [OS1, Theorem 5.8] for example). We recall the algebraic case now and prove Theorem B.

Following [OS1], we write $\mathcal{O}_{\mathrm{alg}}$ for the subcategory of $M \in \mathcal{O}^{\mathfrak{b}}$ such that all the weights of $\mathfrak{t}$ acting on M are algebraic. Such M are generated over $\mathrm{U}(\mathfrak{g})_L$ by highest weight vectors of algebraic weight. We also write $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}} := \mathcal{O}_{\mathrm{alg}} \cap \mathcal{O}^{\mathfrak{p}}$. The $\mathrm{U}(\mathfrak{g})_L$-module structure on an element in $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ naturally extends to a locally analytic action of P, defining a fully faithful embedding $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}} \hookrightarrow \mathcal{O}^{P}$ (see [OS2, Example 2.4(i)]). If U is an L-linear finite-dimensional locally analytic representation of L_P, the generalized Verma module $M_{\mathfrak{p}}(U)$ will lie in $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ if and only if U is an *algebraic* representation of L_P.

Within the category $\mathcal{O}_{\mathrm{alg}}$, Breuil was able to prove an adjunction formula for the Emerton–Jacquet functor (see [Bre2, Théorème 4.3]) in which, rather than locally analytic principal series appearing, one has the Orlik–Strauch representations appearing. The key computation is contained in the following proposition which we will use.

Note that up until now we have worked in $\mathcal{O}^{P}$ but we switch now to elements M in $\mathcal{O}^{P^-}$. Recall that if $M \in \mathcal{O}^{\mathfrak{p}^-}_{\mathrm{alg}} \in \mathcal{O}^{P^-}$ then it has an opposite dual $M^- \in \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}} \subset \mathcal{O}^{P}$.

Proposition 5.3 ([Bre2, Proposition 4.2])**.** *Suppose that $M \in \mathcal{O}^{\mathfrak{p}^-}_{\mathrm{alg}}$ and π is a smooth admissible representation of L_P of finite length. Let V be an L-linear very strongly admissible representation of G. Then there is a canonical isomorphism*

$$\mathrm{Hom}_G(\mathcal{F}^G_{P^-}(M,\pi), V) \simeq \mathrm{Hom}_{(\mathfrak{g},P)}(M^- \otimes \mathcal{C}^{\mathrm{sm}}_c(N_P,\pi), V).$$

Our improvement of Theorem 4.9 is a combination of the previous two results. If U is an irreducible, finite-dimensional, algebraic representation of L_P then it is $\mathfrak{l}_P$-simple and thus the results of Section 4 apply (see Proposition 3.11). If χ is the algebraic highest weight character of U then χ^{-1} is the highest weight for the dual representation U^*. In particular, if Q is the maximal parabolic for U then Q^- is the maximal parabolic for U^*.

Theorem 5.4. *Let U be an irreducible finite-dimensional algebraic representation of L_P with maximal parabolic Q and suppose that π is a finite length smooth admissible representation of L_P admitting a central character such that $\mathrm{Ind}^{L_Q}_{P^- \cap L_Q}(\pi)^{\mathrm{sm}}$ is irreducible. Let V be an L-linear very strongly admissible and $\mathfrak{p}$-acyclic representation of G such that $(U, \pi(\delta_P))$ is non-critical with respect to V. Then the containment*

$$\mathrm{soc}_G \, \mathrm{Ind}^G_{P^-}(U \otimes \pi)^{\mathrm{an}} \subset \mathrm{Ind}^G_{P^-}(U \otimes \pi)^{\mathrm{an}}$$

defines a natural isomorphism

$$\mathrm{Hom}_G(\mathrm{Ind}^G_{P^-}(U \otimes \pi)^{\mathrm{an}}, V) \simeq \mathrm{Hom}_G(\mathrm{soc}_G \, \mathrm{Ind}^G_{P^-}(U \otimes \pi)^{\mathrm{an}}, V).$$

Note that the non-critical hypothesis is with respect to the pair $(U, \pi(\delta_P))$. This is due to our normalizations (see Section 1.6) of the Emerton–Jacquet functor and to help remove the twists by δ_P^{-1} that would appear in each line of the following proof. We comment on the rest of the hypotheses in Theorem 5.4 after the proof.

Proof of Theorem 5.4. Write χ for the highest weight of U so that $U \simeq U_\chi$. Consider the irreducible object $L(\chi^{-1}) \in \mathcal{O}^{P^-}$. By [OS2, Proposition 3.12(b)] we have

$$\mathcal{F}^G_{P^-}(L(\chi^{-1}), \pi) = \mathcal{F}^G_{Q^-}(L(\chi^{-1}), \mathrm{Ind}^{L_Q}_{P^- \cap L_Q}(\pi)^{\mathrm{sm}})$$

(this is the "PQ"-formula mentioned in Remark 5.2). Since $\mathrm{Ind}_{P^-\cap L_Q}^{L_Q}(\pi)^{\mathrm{sm}}$ is irreducible, Theorem 5.1 implies that the right hand side is irreducible.

On the other hand, the exactness of $\mathcal{F}_{P^-}^G(-,\pi)$ and the relation between the Orlik–Strauch representations and parabolic induction implies that

$$\mathrm{soc}_G \mathrm{Ind}_{P^-}^G(U\otimes\pi)^{\mathrm{an}} = \mathrm{soc}_G \mathcal{F}_{P^-}^G(M_{\mathfrak{p}^-}(U^*),\pi) = \mathrm{soc}_G \mathcal{F}_{P^-}^G(L(\chi^{-1}),\pi)$$

As the last term is irreducible, we deduce that we have a natural commuting diagram

$$\begin{array}{ccc} \mathcal{F}_{P^-}^G(L(\chi^{-1}),\pi) & \longrightarrow & \mathcal{F}_{P^-}^G(M_{\mathfrak{p}^-}(U^*),\pi) \\ \simeq\downarrow & & \downarrow\simeq \\ \mathrm{soc}_G \mathrm{Ind}_{P^-}^G(U\otimes\pi)^{\mathrm{an}} & \longrightarrow & \mathrm{Ind}_{P^-}^G(U\otimes\pi)^{\mathrm{an}} \end{array} \tag{23}$$

where the horizontal arrows are inclusions (this also shows that our assumptions force the locally analytic socle to be irreducible).

By Proposition 3.14 we have a natural isomorphism $M_{\mathfrak{p}^-}(U^*)^- \simeq M_{\mathfrak{p}}(U)^\vee$. This also implies that there is a natural isomorphism $L(\chi^{-1})^- \simeq L(\chi)^\vee$ (after taking internal duals both are quotients of $M_{\mathfrak{p}}(U)$) and Proposition 3.15(c) gives a natural isomorphism $L(\chi)^\vee \simeq L(\chi)$. Finally, Proposition 5.3 implies that the quotient map $M_{\mathfrak{p}^-}(U^*) \twoheadrightarrow L(\chi^{-1})$ induces a canonical commuting diagram

$$\begin{array}{ccc} \mathrm{Hom}_G\left(\mathcal{F}_{P^-}^G(M_{\mathfrak{p}^-}(U^*),\pi),V\right) & \xrightarrow{\simeq} & \mathrm{Hom}_{(\mathfrak{g},P)}\left(M_{\mathfrak{p}}(U)^\vee\otimes\mathcal{C}_c^{\mathrm{sm}}(N_P,\pi),V\right) \\ \downarrow & & \downarrow\text{(3a)} \\ \mathrm{Hom}_G\left(\mathcal{F}_{P^-}^G(L(\chi^{-1}),\pi),V\right) & \xrightarrow{\simeq} & \mathrm{Hom}_{(\mathfrak{g},P)}\left(L(\chi)\otimes\mathcal{C}_c^{\mathrm{sm}}(N_P,\pi),V\right) \end{array}$$

Here we have labeled the right hand vertical arrow as (3a), as that is the same map labeled (3a) in the diagram on page 17. Recall we have proved that if (U,π) is non-critical with respect to V then the map (3a) is an isomorphism; the injectivity was proven in Proposition 4.10 and surjectivity in the final step of the proof of Theorem 4.9. We conclude that the restriction map

$$\mathrm{Hom}_G\left(\mathcal{F}_{P^-}^G(M_{\mathfrak{p}^-}(U^*),\pi),V\right) \to \mathrm{Hom}_G\left(\mathcal{F}_{P^-}^G(L(\chi^{-1}),\pi),V\right)$$

is an isomorphism. By the diagram (23), the restriction map

$$\mathrm{Hom}_G(\mathrm{Ind}_{P^-}^G(U\otimes\pi)^{\mathrm{an}},V) \to \mathrm{Hom}_G(\mathrm{soc}_G \mathrm{Ind}_{P^-}^G(U\otimes\pi)^{\mathrm{an}},V)$$

is also an isomorphism. This concludes the proof. □

Remark 5.5. The algebraic assumption on U in Theorem 5.4 could be removed, as long as the $\mathfrak{l}_P$-simple assumption is kept, with a suitable generalization of Proposition 5.3 to the category $\mathcal{O}^P$. We did not attempt to carry that out.

Remark 5.6. Regarding the hypothesis on π, it is sufficient, but not necessary, to assume that $\mathrm{Ind}_{P^-}^G(\pi)^{\mathrm{sm}}$ is irreducible. If $\mathbf{G} = \mathrm{GL}_{n/K}$ then a well-known criterion comes out of the Bernstein–Zelevinsky classification (see [BZ, Theorem 4.2]). For example, if $\pi = \theta_1\otimes\cdots\otimes\theta_n$ is a smooth character of the diagonal torus T then it is necessary and sufficient to assume that $\theta_i(\varpi_K)/\theta_j(\varpi_K) \neq q$ if $i \neq j$, where ϖ_K is any uniformizer of K and the residue field of K has q elements.

A. Generalized Verma modules in the Borel case

The main result, Theorem 3.17, of Section 3.5 was reduced for a general standard parabolic $\mathbf{P}$ to the case of the fixed Borel subgroup $\mathbf{B}$. The corresponding results for $\mathbf{B}$ are proved in [BC, Section 4.1]. Large sections of that reference restrict to the case $\mathbf{G} = \mathrm{GL}_3$ but the reader will notice that this assumption was never practically used in [BC, Section 4.1]. In an effort to make this article self-contained, we include the relevant arguments here. Our main goal is to prove Theorem 3.17(1) in the case where $\mathbf{B} = \mathbf{P}$ (see Proposition A.4). We also recall some notions (specifically, the notion of "strongly linked weights") that were used liberally in the text. We use all the notations of this paper.

To a locally analytic character χ of T we can attach a generalized Verma module $M_{\mathfrak{b}}(\chi)$ as in Section 3.1. As a module over $\mathrm{U}(\mathfrak{g})_L$, it is the classical Verma module $M_{\mathfrak{b}}(d\chi)$. The irreducible constituents of $M_{\mathfrak{b}}(d\chi)$ are well-known to be related to the notion of weights *strongly linked* to $d\chi$. Let us recall this notion, adapted to the characters of T.

For a root α of $\mathfrak{g}$ we denote by $\alpha^\vee$ the corresponding co-root. Each root α is an algebraic weight of $\mathbf{G}$ and we write α also for the corresponding algebraic character (and $\alpha^\vee$ for the corresponding algebraic co-character). Let $\delta : T \to L^\times$ be a locally analytic character. We can pre-compose δ with the co-character $\alpha^\vee : L^\times \to T$ and obtain a canonical character $\langle \delta, \alpha^\vee \rangle$. We say that δ is α-integral if $\langle \delta, \alpha^\vee \rangle$ is of the form $z \mapsto z^m$ for some integer $m \in \mathbf{Z}$. In that case, $\delta + \rho_0$ is also α-integral and we say that δ is α-dominant if $\langle \delta + \rho, \alpha^\vee \rangle$ is a *positive* integer. More generally, if χ is a locally analytic character and α is a root then say that χ is locally α-integral (resp. -dominant) if $\chi = \delta\theta$ where θ is a *smooth* character of T and δ is α-integral (resp. -dominant).

The Weyl group "dot action" from the classical Verma module theory extends to this setting. Recall (see [Hum, Definition 1.8]) if w is an element in the Weyl group with respect to $\mathbf{T}$ and λ is a weight then $w \cdot \lambda := w(\lambda + \rho_0) - \rho_0$. For a root α we denote by s_α the corresponding reflection inside the Weyl group. If χ is a locally α-integral character then it is easy to see that $s_\alpha \cdot d\chi$ differs from $d\chi$ by an algebraic weight.

Definition A.1. *Let α be a root. For a locally α-integral character χ we denote by η the unique algebraic character of T whose weight is $s_\alpha \cdot d\chi - d\chi$ and we define $s_\alpha \cdot \chi := \chi\eta$.*

If χ is locally α-integral and it happens that $s_\alpha \cdot \chi$ is locally α'-integral then we could form the iterated character $s_{\alpha'} s_\alpha \cdot \chi$ in the obvious way. Recall that Φ^+ denotes the fixed set of roots positive with respect to $\mathbf{B}$.

Definition A.2. *Let χ and χ' be two locally analytic characters of T. We write $\chi' \uparrow \chi$ if $\chi' = \chi$ or there exists an $\alpha \in \Phi^+$ such that χ is locally α-dominant and $s_\alpha \cdot \chi = \chi'$. We say that χ' is strongly linked to χ if there exists a sequence of positive roots $\alpha_1, \dots, \alpha_r \in \Phi^+$ such that*

$$\chi' = (s_{\alpha_1} \cdots s_{\alpha_r}) \cdot \chi \uparrow (s_{\alpha_2} \cdots s_{\alpha_r}) \cdot \chi \uparrow \cdots \uparrow s_{\alpha_r} \cdot \chi \uparrow \chi.$$

The relationship $\uparrow$ is not an equivalence relation, but it is reflexive and transitive. We refer the reader to [BC, Section 3.3] for further examples and explanation.

We return to the Verma module $M_{\mathfrak{b}}(\chi)$. Note that χ is irreducible, even one-dimensional, as a representation of $\mathfrak{b}$ and thus all of the results in Sections 3.1-3.4 apply to $U = \chi$. In particular, Proposition 3.15 gives us a unique irreducible quotient in $M_{\mathfrak{b}}(\chi) \twoheadrightarrow L(\chi) \in \mathcal{O}^B$ which appears with multiplicity one as a composition factor. A key point in understanding the rest of the composition factors is the following lemma.

Lemma A.3. *If χ and χ' are two locally analytic characters and $f : M_{\mathfrak{b}}(\chi') \to M_{\mathfrak{b}}(\chi)$ is $\mathrm{U}(\mathfrak{g})_L$-equivariant then f is T-equivariant.*

Proof. This is an explicit computation. Our first remark is that by [Hum, Theorem 4.2] the map f is injective, thus we suppress it from the notation and just view $M_{\mathfrak{b}}(\chi')$ as a $\mathrm{U}(\mathfrak{g})_L$-submodule of $M_{\mathfrak{b}}(\chi)$. We need to show that it is a T-submodule as well.

Write e_χ^+ (resp. $e_{\chi'}^+$) for the highest weight vector in $M_{\mathfrak{b}}(\chi)$ (resp. $M_{\mathfrak{b}}(\chi')$). Since $M_{\mathfrak{b}}(\chi)$ is generated by e_χ^+ over $\mathrm{U}(\mathfrak{n}_B^-)_L$ we can write $e_{\chi'}^+ = X e_\chi^+$ for some $X \in \mathrm{U}(\mathfrak{n}_B^-)_L$. A short computation shows that X is an eigenvector for the bracket action of $\mathfrak{t}$, with weight $d\chi' - d\chi$. But the weights of $\mathfrak{t}$ appearing in $\mathrm{U}(\mathfrak{n}_B^-)_L$ are all algebraic, and agree with the eigensystem for the adjoint action of T on $\mathrm{U}(\mathfrak{n}_B^-)_L$. Thus X is a simultaneous eigenvector for $\{\mathrm{Ad}(t) \mid t \in T\}$ with eigensystem $\chi'\chi^{-1}$. But then Xe_χ^+ is an eigenvector for T with eigenvalues given by χ', and thus $Le_{\chi'}^+$ is a T-stable subrepresentation of $M_{\mathfrak{b}}(\chi)$. Since $M_{\mathfrak{b}}(\chi')$ is generated over $\mathrm{U}(\mathfrak{n}_B^-)_L$ by $e_{\chi'}^+$, we see that $M_{\mathfrak{b}}(\chi')$ is also T-stable as desired. □

Proposition A.4. *If χ and χ' are two locally analytic characters then $L(\chi')$ is a composition factor of $M_{\mathfrak{b}}(\chi)$ in $\mathcal{O}^B$ if and only if χ' is strongly linked to χ. Moreover, all the composition factors of $M_{\mathfrak{b}}(\chi)$ in $\mathcal{O}^B$ are of this form.*

Remark A.5. Proposition A.4 is a more precise version of Theorem 3.17(a) where $\mathbf{B} = \mathbf{P}$, since we specify that *if* χ' is strongly linked to χ *then* $L(\chi')$ actually appears as a composition factor.

Proof of Proposition A.4. Suppose first that χ' is strongly linked to χ. By BGG reciprocity [Hum, Theorem 5.1(a)] $L(\chi')$ is a composition factor of $M_{\mathfrak{b}}(\chi)$ as a $\mathrm{U}(\mathfrak{g})_L$-module and there exists a non-zero $\mathrm{U}(\mathfrak{g})_L$-equivariant morphism $M_{\mathfrak{b}}(\chi') \to M_{\mathfrak{b}}(\chi)$. By Lemma A.3 and Proposition 3.1 the morphism is automatically in $\mathcal{O}^B$ and thus $L(\chi')$ is a subquotient of $M_{\mathfrak{b}}(\chi)$ in $\mathcal{O}^B$.

Suppose that Λ is an irreducible subquotient of $M_{\mathfrak{b}}(\chi)$ in the category $\mathcal{O}^B$. We note that T acts semi-simply on $M_{\mathfrak{b}}(\chi)$ (because T acts semi-simply on $\mathrm{U}(\mathfrak{n}_B^-)_L$ and $M_{\mathfrak{b}}(\chi)$ is generated over $\mathrm{U}(\mathfrak{n}_B^-)_L$ by an eigenvector for T). Choose a highest weight μ for Λ. That is, $\mathrm{U}(\mathfrak{n}_B)_L \cdot \Lambda_\mu = (0)$. The μ-eigenspace Λ_μ is T-stable, thus a sum of one-dimensional spaces, so choose a non-zero vector $v \in \Lambda_\mu$ which is an eigenvector for T. Since $\mathfrak{n}_B$ annihilates v, B acts on v through a character χ'. But then v defines a non-zero $\mathrm{U}(\mathfrak{g})_L$-equivariant map $M_{\mathfrak{b}}(\chi') \to \Lambda$. Since Λ is irreducible as a $(\mathfrak{g}, B)$-module, Λ is an irreducible quotient of $M_{\mathfrak{b}}(\chi')$. But then by Proposition 3.15 we have $\Lambda \simeq L(\chi')$. Finally, by [Hum, Theorem 5.2(b)] we see immediately that χ' is strongly linked to χ and we are done. □

With the composition series for the generalized Verma modules clarified by Proposition A.4, we can state a slightly more precise version of Theorem 4.9 which may be easier to reference. An analog of it was first proven as [BC, Theorem 4.1].

Theorem A.6. *Suppose that χ is a locally analytic L-valued character and V is an L-linear very strongly admissible, $\mathfrak{b}$-acyclic, representation of G. If $J_P(V)^{T=\chi'} = (0)$ for all strongly linked characters $\chi' \neq \chi$ then there is a canonical isomorphism*

$$\mathrm{Hom}_G(\mathrm{Ind}_{B^-}^G(\chi\delta_B^{-1})^{\mathrm{an}}, V) \simeq J_P(V)^{T=\chi}.$$

The reader can check that this follows immediately from Theorem 4.9, Theorem 3.17 and the additional information given by Proposition A.4.

References

[BC] J. Bergdall and P. Chojecki, "Ordinary representations and companion points for U(3) in the indecomposable case", Preprint (2014).

[BZ] I. N. Bernstein and A. V. Zelevinsky, "Induced representations of reductive $\mathfrak{p}$-adic groups. I", *Ann. Sci. École Norm. Sup. (4)*, 10(4):441–472 (1977).

[Bor] A. Borel, *Linear algebraic groups*, Vol. 126 of *Graduate Texts in Mathematics*. Springer–Verlag, New York, 2nd ed. (1991).

[Bre1] C. Breuil, "Vers le socle localement analytique pour GL_n, I", Preprint (2014).

[Bre2] C. Breuil, "Vers le socle localement analytique pour GL_n, II", *Math. Ann.*, 361(3-4):741–785 (2015).

[BE] C. Breuil and M. Emerton, "Representations p-adiques ordinaires de $\mathrm{GL}_2(\mathbf{Q}_p)$ et compatibilite local-global", *Asterisque* 331:255–315 (2010).

[BHS] C. Breuil, E. Hellmann and B. Schraen, "Une interpretation modulaire de la variete trianguline", Preprint (2015).

[BH] C. Breuil and F. Herzig, "Ordinary representations of $G(\mathbf{Q}_p)$ and fundamental algebraic representations", to appear in *Duke Math. J.*

[CE] F. Calegari and M. Emerton, "Completed cohomology—a survey". In *Non-abelian fundamental groups and Iwasawa theory*, Volume 393 of *London Math. Soc. Lecture Note Ser.*, pages 239–257. Cambridge Univ. Press (2012).

[CM] R. Coleman and B. Mazur, "The eigencurve", In *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, volume 254 of *London Math. Soc. Lecture Note Ser.*, pages 1–113. Cambridge Univ. Press, Cambridge (1998).

[Di] Y. Ding, *Formes modulaires p-adiques sur les courbes de Shimura unitaires et comparibilité local-global.* Ph.D. thesis, Université Paris-Sud (2014).

[Eme1] M. Emerton, "On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms", *Invent. Math.*, 164(1):1–84 (2006).

[Eme2] M. Emerton, "Jacquet modules of locally analytic representations of p-adic reductive groups I. Construction and first properties", *Ann. Sci. E.N.S. (4)*, 39(5):775–839 (2006).

[Eme3] M. Emerton, "Jacquet modules of locally analytic representations of p-adic reductive groups II. The relation to parabolic induction", to appear in *J. Institut Math. Jussieu.*

[Eme4] M. Emerton, "Locally analytic vectors in representations of locally p-adic analytic groups", to appear in *Memoirs of the AMS* (2011).

[HL] R. Hill and D. Loeffler, "Emerton's Jacquet functors for non-Borel parabolic subgroups", *Doc. Math.* 16:1–31 (2011).

[Hum] J. Humphreys, *Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$*, volume 94 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI (2008).

[OS1] S. Orlik and M. Strauch, "On Jordan–Hölder series of some locally analytic representations". *J. Amer. Math. Soc.* 28(1):99–157 (2015).

[OS2] S. Orlik and M. Strauch, "Category $\mathcal{O}$ and locally analytic representations", Preprint (2014).

[ST1] P. Schneider and J. Teitelbaum, "U($\mathfrak{g}$)-finite locally analytic representations", *Represent. Theory*, 5:111–128 (electronic), (2001). With an appendix by D. Prasad.

[ST2] P. Schneider and J. Teitelbaum, "Duality for admissible locally analytic representations", Represent. Theory 9, 297–326 (electronic), (2005).

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