ON NON-ABELIAN LUBIN-TATE THEORY AND ANALYTIC COHOMOLOGY

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Abstract. We prove that the p-adic local Langlands correspondence for $GL_2(\mathbb{Q}_p)$ appears in the étale cohomology of the Lubin-Tate tower at infinity. We use global methods using recent results of Emerton on the local-global compatibility and hence our proof applies to local Galois representations which come via a restriction from a global pro-modular Galois representations. We also discuss a folklore conjecture which states that the p-adic local Langlands correspondence appears in the de Rham cohomology of the Lubin-Tate tower (Drinfeld tower). We show that a study of the de Rham cohomology for perfectoid spaces reduces to a study of the analytic cohomology and we state a natural conjecture related to it.

Contents

1. Introduction

The p-adic Langlands program, started by Christophe Breuil and developed largely by Laurent Berger, Pierre Colmez, Matthew Emerton and Mark Kisin, has as a goal to establish a

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correspondence between p-adic Galois representations and representations of p-adic reductive groups on p-adic Banach spaces. It has (and will have) many applications, for example the Fontaine-Mazur conjecture (see [Em2] or [CS]). Unfortunately, at the moment the p-adic correspondence is constructed (mostly by Colmez) only for $\mathrm{GL}_2(\mathbb{Q}_p)$ and it seems hard to generalize it to other groups because of many algebraic obstacles.

Hence it seems natural to try to find the correspondence in the appropiate cohomology groups as was done for the classical Langlands correspondence. We are interested in the p-adic completed and analytic cohomologies of Shimura varieties and Rapoport-Zink spaces, which are natural objects to consider in the context of the Langlands program. This paper might be seen as a sequel to [Cho], where we have studied the mod p étale cohomology of the Lubin-Tate tower. Here we turn to the study of the p-adic completed and analytic cohomologies.

There are two goals which we want to accomplish in this short paper. The first one is to show a result analogous to the one obtained in [Cho], namely to show that the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ appears in the étale cohomology of the Lubin-Tate tower at infinity. The methods we use are partly those of [Cho] (localisation at a supersingular representation; use of the local-global compatibility of Emerton), though we approach them differently by working in the setting of adic spaces (we have worked with Berkovich spaces in [Cho]). This gives us more freedom as we can work directly at the infinite level (modular curves at the infinite level; Lubin-Tate tower at the infinite level) thanks to the work of Scholze on perfectoid spaces ([Sch1], [Sch2], [SW]). In this way, we do not need anymore to pass to the limit in the cohomology, as working at the infinite level is the same as working with the completed cohomology (see Chapter IV of [Sch2]). We prove our main result (Theorem 4.3) for local Galois representations ρ_p which are restrictions of some global pro-modular (a notion from [Em2]) representations ρ and such that the mod p reduction $\bar{\rho}_p$ is absolutely irreducible. We need these assumptions in order to be able to use the main result of [Em2].

The second goal of this text is to discuss the folklore conjecture which states that the p-adic local Langlands correspondence appears in the de Rham cohomology of the Drinfeld tower. As far as we know, this conjecture is not stated anywhere explicitely in the literature, though there was some work done towards it. The reader should consult [Schr] for some partial progress at the 0-th level of the tower. Thanks to the work of Scholze-Weinstein ([SW]) we can work directly at the infinite level which we do. Moreover, because of the duality of Rapoport-Zink spaces at the infinite level (which goes back to Faltings; see Section 7 of [SW]), we know that the Drinfeld space at infinity $\mathcal{M}_{Dr,\infty}$ is isomorphic to the Lubin-Tate space at infinity $\mathcal{M}_{LT,\infty}$ and hence we can consider only the Lubin-Tate tower which is easier to relate to modular curves. Nevertheless, our result (Theorem 4.3) work equally well for $\mathcal{M}_{Dr,\infty}$.

As to the folklore conjecture, we give a short argument at the beginning of Section 4, which explains why the de Rham cohomology of $\mathcal{M}_{LT,\infty}$ simplifies greatly. The reason is that for any perfectoid space X (hence for $\mathcal{M}_{LT,\infty}$ after [SW]) the cohomology groups of j-th differentials $H^i(X, \Omega^j_X)$ vanishes for $j > 0$ and any i. This reduces a study of the de Rham cohomology to a study of the cohomology of the structure sheaf (which we refer to as the analytic cohomology - with topology defined by open subsets) which should be a good substitute for the de Rham cohomology in the setting of perfectoid spaces. We state the folklore conjecture for the analytic cohomology in the last section.

At the end we remark that a problem with the de Rham cohomology for perfectoid spaces, if one would like to define it in some meaningful way, is the lack of finiteness result. We should mention the work of Cais ([Cai]), where the author consider integral structures on the de Rham cohomology of curves. The aim is to p-adically complete the de Rham cohomology of the tower of modular curves, as was done with the étale cohomology by Emerton ([Em1]). It seems interesting to determine what one would get by applying his construction at each finite level and then passing to the limit and how it would relate to the de Rham cohomology of the modular curve at the infinite level.

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2. Modular curve at infinity

In this section we review the geometric background which we use. We start by recalling basic definitions about adic spaces and perfectoid spaces. Then we describe modular curves (and their compactifications) at the infinite level and we deal with the ordinary locus and the supersingular locus.

2.1. **Adic spaces.** We recall here a basic material about adic spaces. For a careful study see [Hu2].

Let R be a topological ring. We call R an adic ring if there exists an ideal I of R such that $\{I^n\}_{n\in\mathbb{N}}$ form a fundamental system of neighbourhoods of $0 \in R$. I is called an ideal of definition of R.

R is called f-adic if there exists an open subring R_0 of R such that R_0 with the subspace topology is adic and has a finitely generated ideal of definition. In this case, every open subring of R whose subspace topology is adic is called a ring of definition of R. Every ring of definition of R has a finitely generated ideal of definition.

A subset $S \subset R$ is bounded if for every neighbourhood U of 0 in R, there exists another neighbourhood V of 0 with $VS \subset U$. An element $f \in R$ is power-bounded if $\{f^n\}_{n\in\mathbb{N}}$ is bounded. We write R° for the subring of power-bounded elements of R. Denote by $R^{\circ\circ} \subset R^\circ$ the ideal of topologically nilpotent elements.

An affinoid ring is a pair (R, R^+) with R an f-adic ring and $R^+ \subset R$ an open subring which is integrally closed and contained in R°. Morphisms between affinoid rings (R, R^+) and (S, S^+) are continuous ring homomorphisms $R \to S$ sending R^+ into S^+.

A continuous valuation on R is a multiplicative map $|\cdot| : R \to \Gamma \cup \{0\}$, where Γ is a linearly ordered abelian group (written multiplicatively) such that $|0| = 0$, $|1| = 1$, $|x+y| \leq max(|x|, |y|)$ and for all $\gamma \in \Gamma$, the set $\{x \in R : |x| < \gamma\}$ is open. Two valuations $|\cdot|_i : R \to \Gamma_i \cup \{0\}$ $(i = 1, 2)$ are equivalent if there are subgroups $\Gamma_i' \subset \Gamma_i$ containing the image of $|\cdot|_i$ for $i = 1, 2$ and an isomorphism of ordered groups $i : \Gamma_1' \to \Gamma_2'$ such that $i(|f|_1) = |f|_2$ for all $f \in R$.

Let (R, R^+) be an affinoid ring. We define the adic spectrum $X = \mathrm{Spa}(R, R^+)$ as the set of equivalence classes of continuous valuations on R which satisfy $|R^+| \leq 1$. For $x \in X$ and $f \in R$, we write $|f(x)|$ for the image of f under the continuous valuation $|\cdot|_x$ corresponding to x.

Let $f_1, ..., f_n$ be elements of R such that $(f_1, ..., f_n)R$ is open in R and let $g \in R$. We define the subset

$$X_{f_1,...,f_n;g} = \{x \in X | |f_i(x)| \leq |g(x)| \neq 0, i = 1, ..., n\}$$

We call finite intersections of such subsets rational. We give X the topology generated by its rational subsets.

One can define a presheaf of complete topological rings $\mathcal{O}_X$ on X, together with a subpresheaf $\mathcal{O}_X^+$. They can be characterized by the following universal property. Let $U \subset X$ be a rational subset. Then there is a complete affinoid ring $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ with a map $(R, R^+) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ such that

$$\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to \mathrm{Spa}(R, R^+)$$

factors through U and for any complete affinoid ring (S, S^+) with a map $(R, R^+) \to (S, S^+)$ for which $\mathrm{Spa}(S, S^+) \to \mathrm{Spa}(R, R^+)$ factors through U, the map $(R, R^+) \to (S, S^+)$ extends to a map

$$(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to (S, S^+)$$

in a unique way.

One can check that $\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \simeq U$ and that the valuations on (R, R^+) associated to $x \in U$ extend to $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$. In particular, they extend further to $\mathcal{O}_{X,x}$. Also we have

$$\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) | |f(x)| \leq 1 \forall_{x \in U}\}$$

We remark that the presheaf $\mathcal{O}_X$ is not a sheaf in general, though one can prove that it is a sheaf when R has a noetherian ring of definition, is strongly noetherian or perfectoid.

An adic space is a locally ringed space which is locally isomorphic to an affinoid adic space

$$(\mathrm{Spa}(R, R^+), \mathcal{O}_{\mathrm{Spa}(R,R^+)}, \mathcal{O}^+_{\mathrm{Spa}(R,R^+)})$$

Actually, one should keep track also of valutations, but we ignore this issue here. See [Hu2] or [Sch1] for details. For a more general definition of an adic space see [SW].

Let X_i be a filtered inverse system of adic spaces with quasicompact and quasiseparated transition maps, let X be an adic space and let $f_i : X \to X_i$ be a compatible set of morphisms. We write

$$X \sim X_i$$

if the map of underlying topological spaces $|X| \to \varprojlim |X_i|$ is a homeomorphism, and if there is an open cover of X by affinoids $\mathrm{Spa}(R, R^+) \subset X$ such that the map

$$\varinjlim_{\mathrm{Spa}(R_i, R_i^+) \subset X_i} R_i \to R$$

has dense image, where the direct limit runs over all open affinoids $\mathrm{Spa}(R_i, R_i^+) \subset X_i$ over which $\mathrm{Spa}(R, R^+) \subset X \to X_i$ factors.

2.2. **Perfectoid spaces.** Let us also shortly recall a basic material about perfectoid spaces. We refer the reader to [Sch1] for details.

A perfectoid field K is a complete non-archimedean field with a non-discrete valuation and such that the Frobenius map $\mathcal{O}_K/p \to \mathcal{O}_K/p$ is surjective. A perfectoid K-algebra is a complete Banach K-algebra A such that the subalgebra A° of power-bounded elements is bounded in A and the Frobenius map $A^\circ/p \to A^\circ/p$ is surjective. A perfectoid affinoid $(K, \mathcal{O}_K)$-algebra is an affinoid $(K, \mathcal{O}_K)$-algebra (A, A^+) such that A is a perfectoid K-algebra.

A perfectoid space over $\mathrm{Spa}(K, \mathcal{O}_K)$ is an adic space which is locally isomorphic to $\mathrm{Spa}(A, A^+)$ for a perfectoid affinoid $(K, \mathcal{O}_K)$-algebra (A, A^+).

Perfectoid spaces are particularly nice adic spaces which are not of finite type. Their construction is well-suited for 'extracting infinitely many p-power roots'. One can prove (see [Sch1]) that both $\mathcal{O}_X$ and $\mathcal{O}_X^+$ are sheaves for a perfectoid space X. One of the crucial properties of perfectoid spaces is the fact that one can pass easily between characteristic 0 and characteristic p.

If K is a perfectoid field of characteristic 0, then we define its tilt by $K^\flat = \varprojlim_{x \mapsto x^p} K$. Scholze proves that $K^\flat$ is again perfectoid but lives in characteristic p (residue characteristic of K). This process generalizes to perfectoid affinoids (R, R^+) (we associate to (R, R^+) its tilt $(R^\flat, R^{+\flat})$) and hence also to general perfectoid spaces (X over $\mathrm{Spa}(K, \mathcal{O}_K)$ tilts to $X^\flat$ over $\mathrm{Spa}(K^\flat, \mathcal{O}_{K^\flat})$).

Tilting is a useful process which does not change the underlying topological space neither the étale topos: $|X|$ is homeomorphic to $|X^\flat|$ and the étale topos of X is equivalent to the étale topos of $X^\flat$. This allows to prove many properties about perfectoid spaces in characteristic 0 by working purely in positive characteristic.

2.3. **Geometry of modular curves.** We define open modular curves over $\mathbb{C}$ for an open compact subset $K \subset \mathrm{GL}_2(\mathbb{A}_f)$ by

$$Y(K) = \mathrm{GL}_2(\mathbb{Q}) \backslash (\mathbb{C} \backslash \mathbb{R}) \times \mathrm{GL}_2(\mathbb{A}_f)/K$$

One can find an algebraic model over a local field (a finite extension of $\mathbb{Q}_p$). We fix some complete and algebraically closed extension C of $\mathbb{Q}_p$. Let $\mathcal{O}_C$ be the ring of integers of C. We consider modular curves as adic spaces over $\mathrm{Spa}(C, \mathcal{O}_C)$ which we may do after base-changing each $Y(K)$.

We let $X(K)$ be the Katz-Mazur compactification of $Y(K)$, which we also consider as an adic space over $\mathrm{Spa}(C, \mathcal{O}_C)$. We will work with modular curves at the infinite level.

Proposition 2.1. *For any sufficiently small level $K^p \subset \mathrm{GL}_2(\mathbb{A}_f^p)$ there exist adic spaces $Y(K^p)$ and $X(K^p)$ over $\mathrm{Spa}(C, \mathcal{O}_C)$ such that*

$$Y(K^p) \sim \varprojlim_{K_p} Y(K_p K^p)$$

$$X(K^p) \sim \varprojlim_{K_p} X(K_p K^p)$$

where K_p runs over open compact subgroups of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Proof. See Theorem III.1.2 in [Sch2]. □

By passing again to the limit we get

Proposition 2.2. *There exist adic spaces* Y *and* X *over* $\mathrm{Spa}(C, \mathcal{O}_C)$ *such that*

$$Y \sim \varprojlim_{K^p} Y(K^p) \sim \varprojlim_{K_pK^p} Y(K_pK^p)$$

$$X \sim \varprojlim_{K^p} X(K^p) \sim \varprojlim_{K_pK^p} X(K_pK^p)$$

where K^p *runs over sufficiently small compact open subgroups of* $\mathrm{GL}_2(\mathbb{A}_f^p)$ *and* K_p *runs over open compact subgroups of* $\mathrm{GL}_2(\mathbb{Q}_p)$.

At each finite level we define the supersingular locus $Y(K_pK^p)_{ss}$ and the ordinary locus $Y(K_pK^p)_{ord}$ of $Y(K_pK^p)$. Recall that there is a reduction map from an adic space to the special fiber of the corresponding formal model. Indeed, if x is a point of $\mathrm{Spa}(A, A^+)$, then it gives rise to a continuous character $\chi_x : A \to k(x)$, where $k(x)$ is the quotient field associated to x (see p.461 in [Hu1]). By composing it with the projection to the residue field $\overline{k(x)}$ of $k(x)$ we get a character $\bar{\chi}_x : A/I \to \overline{k(x)}$, where I is an ideal of definition of A. The kernel of $\bar{\chi}_x$ is a prime ideal in $\mathrm{Spec}\, A/I$ as wanted.

We define $Y(K_pK^p)_{ss}$ as the inverse image under the reduction of the set of supersingular points in the special fiber of $Y(K_pK^p)$. It is an open subspace of $Y(K_pK^p)$. We define $Y(K_pK^p)_{ord}$ as the closure of the inverse image under the reduction of the ordinary locus in the special fiber. This is the complement of $Y(K_pK^p)_{ss}$ and hence a closed subspace of $Y(K_pK^p)$. We define similarly the supersingular locus $X(K_pK^p)_{ss}$ and the ordinary locus $X(K_pK^p)_{ord}$ of $X(K_pK^p)$.

Again we notice that at infinity we can also define both the supersingular and the ordinary locus. The reader may consult a discussion in [Sch2] which appears after Theorem III.1.2.

Proposition 2.3. *There exist adic spaces* Y_{ss}, Y_{ord} *and* X_{ss}, X_{ord} *over* $\mathrm{Spa}(C, \mathcal{O}_C)$ *such that*

$$Y_{ss} \sim \varprojlim_{K_pK^p} Y(K_pK^p)_{ss}$$

$$Y_{ord} \sim \varprojlim_{K_pK^p} Y(K_pK^p)_{ord}$$

and similarly for X_{ss} *and* X_{ord}. *Here* K^p *runs over sufficiently small compact open subgroups of* $\mathrm{GL}_2(\mathbb{A}_f^p)$ *and* K_p *runs over open compact subgroups of* $\mathrm{GL}_2(\mathbb{Q}_p)$.

We let

$$j : X_{ss} \hookrightarrow X$$

denote the open immersion and we put

$$i : X_{ord} \to X$$

For any injective étale sheaf I on X we have an exact sequence of global sections

$$0 \to \Gamma_{X_{ord}}(X, I) \to \Gamma(X, I) \to \Gamma(X_{ss}, j^*I) \to 0$$

which gives rise to the exact sequence of cohomology for any étale sheaf F on X (take an injective resolution $I^\bullet$ of F and apply the above exact sequence to it)

$$... \to H^0(X_{ss}, j^*F) \to H^1_{X_{ord}}(X, F) \to H^1(X, F) \to H^1(X_{ss}, j^*F) \to ...$$

By specialising F to a constant sheaf $\mathbb{Z}/p^s\mathbb{Z}$ ($s > 0$) we get an exact sequence

$$... \to H^0(X_{ss}, \mathbb{Z}/p^s\mathbb{Z}) \to H^1_{X_{ord}}(X, \mathbb{Z}/p^s\mathbb{Z}) \to H^1(X, K) \to H^1(X_{ss}, \mathbb{Z}/p^s\mathbb{Z}) \to ...$$

We can obtain an analogous exact sequence for analytic cohomology which we review later. In what follows we work with the p-adic completed cohomology, introduced by Emerton in [Em1]. We define for a finite extension K of $\mathbb{Q}_p$

$$H^1(X, K) = \left(\varprojlim_s H^1_{et}(X, \mathbb{Z}/p^s\mathbb{Z}) \right) \otimes_{\mathbb{Z}_p} K$$

Using the fact that $X \sim \varprojlim_{K_pK^p} X(K_pK^p)$, we have

$$H^1(X, K) = \left(\varinjlim_{K^p} \varprojlim_s \varinjlim_{K_p} H^1_{et}(X(K_pK^p), \mathbb{Z}/p^s\mathbb{Z}) \right) \otimes_{\mathbb{Z}_p} K$$

which is precisely the p-adic completed cohomology. We use similar definitions for X_{ss} and X_{ord}. This is indeed the cohomology theory as one can see from a direct check in our case or by a comparison isomorphism with the analytic cohomology (Theorem IV.2.1 in [Sch2]). Hence we have an exact sequence

$$... \to H^0(X_{ss}, K) \to H^1_{X_{ord}}(X, K) \to H^1(X, K) \to H^1(X_{ss}, K) \to ...$$

which will be crucial to us later on.

2.4. **Ordinary locus.** We give a decomposition of the ordinary locus, which implies that representations arising from the cohomology are induced from a Borel subgroup.

Proposition 2.4. *There exist adic spaces $\mathcal{C}_a$ over $\mathrm{Spa}(C, \mathcal{O}_C)$ where $a \in \mathbb{P}^1(\mathbb{Q}_p)$ such that*

$$X_{ord} = \coprod_a \mathcal{C}_a$$

The action of $\mathrm{GL}_2(\mathbb{Q}_p)$ is compatible with this decomposition. In particular, the étale (and also analytic) cohomology of X_{ord} is induced from a Borel subgroup $B(\mathbb{Q}_p)$ of upper-triangular matrices in $\mathrm{GL}_2(\mathbb{Q}_p)$

$$H^i_{X_{ord}}(X, F) = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} W(F)$$

where F is an étale sheaf on X_{ord} and $W(F)$ is a certain cohomology space defined below in the proof which depends on F and admits an action of $B(\mathbb{Q}_p)$.

Proof. This is a classical and well-known result (see Section 2.2 of [Cho]), but we shall give it a short proof using recent results of Scholze and the fact that we are working at the infinite level. In [Sch2] (see Theorem III.1.2 and a discussion after it), Scholze has showed an existence of a natural $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant period map (the Hodge-Tate period map)

$$\pi_{HT} : X \to (\mathbb{P}^1)^{ad}$$

where $(\mathbb{P}^1)^{ad}$ is the adic projective line over $\mathrm{Spa}(C, \mathcal{O}_C)$. Moreover, one has $X_{ord} = \pi_{HT}^{-1}(\mathbb{P}^1(\mathbb{Q}_p))$. Let us define for any $a \in \mathbb{P}^1(\mathbb{Q}_p)$ an adic space $\mathcal{C}_a = \pi_{HT}^{-1}(a)$. Because π_{HT} is $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant, spaces $\mathcal{C}_a$ are as required.

For the second statement, let $\infty = \binom{1}{0} \in \mathbb{P}^1(\mathbb{Q}_p)$. The stabilizer of ∞ is equal to the Borel subgroup $B(\mathbb{Q}_p)$ of upper-triangular matrices in $\mathrm{GL}_2(\mathbb{Q}_p)$. Hence $W(F)$ of the statement is equal to $H^i_{\mathcal{C}_\infty}(X, F)$. □

2.5. **Supersingular locus.** Let us denote by $\mathcal{M}_{LT,K_p}$ the Lubin-Tate space for $\mathrm{GL}_2(\mathbb{Q}_p)$ at the level K_p, where K_p is a compact open subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$. See Section 6 of [SW] for a definition. We just recall that this is a deformation space for p-divisible groups with an additional data and it is a local analogue of modular curves. We view it as an adic space over $\mathrm{Spa}(C, \mathcal{O}_C)$.

Once again, we would like to pass to the limit and work with the space at infinity.

Proposition 2.5. *There exists a perfectoid space $\mathcal{M}_{LT,\infty}$ over $\mathrm{Spa}(C, \mathcal{O}_C)$ such that*

$$\mathcal{M}_{LT,\infty} \sim \varprojlim_{K_p} \mathcal{M}_{LT,K_p}$$

where K_p runs over compact open subgroups of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Proof. This is Theorem 6.3.4 from [SW]. One defines $\mathcal{M}_{LT,\infty}$ as a deformation functor of p-divisible groups with a trivialization of Tate modules. □

To compare X and $\mathcal{M}_{LT,\infty}$ (hence their cohomology groups) we use the p-adic uniformisation of Rapoport-Zink at the infinite level. Let us denote by D the quaternion algebra over $\mathbb{Q}$ which is ramified exactly at p and ∞. The p-adic uniformisation of Rapoport-Zink states

Proposition 2.6. *We have an isomorphism of adic spaces*

$$X_{ss} = D^\times(\mathbb{Q}) \backslash \left(\mathcal{M}_{LT,\infty} \times \mathrm{GL}_2(\mathbb{A}_f^p) \right)$$

This isomorphism is equivariant with respect to the action of the prime-to-p Hecke algebra.

For the proof of the above proposition see [RZ].

3. On admissible representations

Having established all the neccesary geometric results, we now pass to the results about representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. We review and prove some facts about Banach admissible representations. Then we recall recent results of Paskunas which allow us to consider the localisation functor.

3.1. **General facts and definitions.** We start with general facts about admissible representations. In our definitions, we will follow [Em3]. Let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, a uniformiser ϖ and the residue field k. Let $C(\mathcal{O})$ denote the category of complete Noetherian local $\mathcal{O}$-algebras having finite residue fields. Let us consider $A \in C(\mathcal{O})$. We let G be any connected reductive group over $\mathbb{Q}_p$.

Definition 3.1. *Let V be a representation of G over A. A vector $v \in V$ is smooth if v is fixed by some open subgroup of G and v is annihilated by some power $\mathfrak{m}^i$ of the maximal ideal of A. Let V_{sm} denote the subset of smooth vectors of V. We say that a G-representation V over A is smooth if $V = V_{sm}$.*

A smooth G-representation V over A is admissible if $V^H[\mathfrak{m}^i]$ (the $\mathfrak{m}^i$-torsion part of the subspace of H-fixed vectors in V) is finitely generated over A for every open compact subgroup H of G and every $i \geq 0$.

Definition 3.2. *We say that a G-representation V over A is ϖ-adically continuous if V is ϖ-adically separated and complete, $V[\varpi^\infty]$ is of bounded exponent, $V/\varpi^i V$ is a smooth G-representation for any $i \geq 0$.*

Definition 3.3. *A ϖ-adically admissible representation of G over A is a ϖ-adically continuous representation V of G over A such that the induced G-representation on $(V/\varpi V)[\mathfrak{m}]$ is admissible smooth over $A/\mathfrak{m}$.*

This definition implies that for every $i \geq 0$, the G-representation $V/\varpi^i V$ is smooth admissible. See Remark 2.4.8 in [Em3].

Remark 3.4. *The ϖ-adically admissible representations are called Banach admissible in* [ST]*, where authors work over a field. We will use this latter notion most of the time even in the context of rings.*

Proposition 3.5. *The category of admissible K-Banach representations is abelian.*

Proof. The category is anti-equivalent to the category of finitely generated augmented modules over certain completed group rings. See Proposition 2.4.11 in [Em3]. □

Now, we will prove an analogue of Lemma 13.2.3 from [Bo] in the $l = p$ setting. We will later apply this lemma to the cohomology of the ordinary locus to force its vanishing after localisation at a supersingular representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. We have proved it already in the mod p setting as Lemma 3.3 in [Cho].

Lemma 3.6. *For any smooth admissible representation (π, V) of the parabolic subgroup $P \subset G$ over A, the unipotent radical U of P acts trivially on V.*

Proof. Let L be a Levi subgroup of P, so that $P = LU$. Let $v \in V$ and let $K_P = K_L K_U$ be a compact open subgroup of P such that $v \in V^{K_P}$. We choose an element z in the centre of L such that:

$$z^{-n} K_P z^n \subset ... \subset z^{-1} K_P z \subset K_P \subset z K_P z^{-1} \subset ... \subset z^n K_P z^{-n} \subset ...$$

and $\bigcup_{n \geq 0} z^n K_P z^{-n} = K_L U$. For every n and m, modules $V^{z^{-n} K_P z^n}[\mathfrak{m}^i]$ and $V^{z^{-m} K_P z^m}[\mathfrak{m}^i]$ are of the same length for every $i \geq 0$, as they are isomorphic via $\pi(z^{n-m})$ and hence we have not only an isomorphism but an equality $V^{z^{-n} K_P z^n}[\mathfrak{m}^i] = V^{z^{-m} K_P z^m}[\mathfrak{m}^i]$. By smoothness, for every $x \in V$, there exists i such that $x \in V[\mathfrak{m}^i]$. Thus we have $x \in V^{K_P}[\mathfrak{m}^i] = V^{z^{-n} K_P z^n}[\mathfrak{m}^i] = V^{K_L U}[\mathfrak{m}^i]$ which is contained in $V^U[\mathfrak{m}^i]$. □

Lemma 3.7. *For any ϖ-adically admissible representation (π, V) of the parabolic subgroup $P \subset G$ over A, the unipotent radical U of P acts trivially on V.*

Proof. By the remark above, each $V/\varpi^i V$ is admissible, and hence the preceding lemma applies, so that U acts trivially on each $V/\varpi^i V$. But $V = \varprojlim_i V/\varpi^i V$, hence U acts trivially on V. □

Later on, we will need the following result.

Lemma 3.8. *Let $V = \mathrm{Ind}_P^G W$ be a parabolic induction. If V is a Banach admissible representation of G over K, then W is a Banach admissible representation of P over K.*

Proof. This follows from Theorem 4.4.6 in [Em3]. □

3.2. **Localisation functor.** Let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and the residue field k. Let π be a supersingular representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over k. Recall that supersingular representations correspond to irreducible two-dimensional Galois representations under the local Langlands correspondence modulo p. See [Be] for details.

In [Pa], Paskunas has proved the following result (Proposition 5.32)

Proposition 3.9. *We have a decomposition:*

$$\mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p),\xi}(K) = \mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p),\xi}(K)_{(\pi)} \oplus \mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p),\xi}(K)^{(\pi)}$$

where $\mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p),\xi}(K)$ *is the (abelian) category of Banach admissible K-representations admitting a central character ξ,* $\mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p),\xi}(K)_{(\pi)}$ *(resp.* $\mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p),\xi}(K)^{(\pi)}$*) is the subcategory of it consisting of representations Π such that for every open bounded* $\mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant lattice Θ in Π, all irreducible subquotients of $\Theta \otimes_{\mathcal{O}} k$ are (resp. are not) isomorphic to π.*

We denote the projection

$$\mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p),\xi}(K) \mapsto \mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p),\xi}(K)_{(\pi)}$$

by

$$V \mapsto V_{(\pi)}$$

and we refer to it as the localisation functor with respect to π. We remark that the assumption on the existence of a central character ξ is not needed by the results of [DS]. We will ignore this issue in what follows.

4. p-adic Langlands correspondence and analytic cohomology

In this section we show that the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ appears in the étale cohomology of the Lubin-Tate tower at infinity. We also state a conjecture about the analytic cohomology of the Lubin-Tate perfectoid. Let us explain, why we do not work with the de Rham cohomology as would the folklore conjecture suggest (to be precise: we do, but we work only with the structure sheaf as all the other differentials vanish as we show below). The reason for that is that there are no good finiteness results for de Rham cohomology of adic spaces which are not of finite type (as our Lubin-Tate perfectoid $\mathcal{M}_{LT,\infty}$). Moreover, it seems that the de Rham cohomology does not suit well perfectoid spaces. Indeed, $H^i(X, \Omega^j_X)$ is zero for any perfectoid space X and sheaves of differentials Ω^j_X, $j > 0$. To see this, it is enough to prove it for affinoid perfectoids $X = \mathrm{Spa}(R, R^+)$. We can further reduce ourselves to the case $i = 0$ by using the Cech complex associated to some rational covering of X (which will by a covering by affinoid perfectoids by Corollary 6.8 of [Sch1]). Hence, we have to show that global sections of Ω^j_X are zero. As X is perfectoid we can pass to the tilt $X^\flat$ in characteristic p. As the étale topoi of X and $X^\flat$ are equivalent (see [Sch1]), it is enough to prove $H^0(X^\flat, \Omega^j_{X^\flat}) = 0$ for $j > 0$. Now it is enough to observe that if $dx \in \Omega^1_{X^\flat}$ and y is such that $x = y^p$ (we can always find such y as $X^\flat$ is perfectoid), then $dx = pdy^{p-1} = 0$ as we are in characteristic p. We conclude similarly for higher differentials. Let us remark that this reasoning also implies that sheaves Ω^j_X are zero on a perfectoid space X for $j > 0$. It is enough to check it at stalks where we have $\Omega^j_{X,x} = \varinjlim_{x \in U} \Omega^j_X(U)$ and U runs over rational affinoid subsets of X containing x. As such subsets are perfectoid (Corollary 6.8 of [Sch1]) we have $\Omega^j_X(U) = 0$ and hence the result.

As $\mathcal{M}_{LT,\infty}$ is a perfectoid space by [SW], the above reasoning applies, showing that de Rham cohomology of $\mathcal{M}_{LT,\infty}$ reduces to the study of the cohomology with values in the structure sheaf. This is exactly the analytic cohomology we consider. By using recent results of Scholze, it seems natural to work with the analytic cohomology (i.e. topology defined by open subsets). We review this below. We believe that the 'folklore conjecture' should be understood as the statement that the p-adic local Langlands correspondence appears in the analytic cohomology of the appropiate Rapoport-Zink space at infinity. We also remark that the same applies to Shimura varieties at the infinite level, which are perfectoid spaces by [Sch2].

4.1. **p-adic Langlands correspondence.** For this section we refer the reader to [Be] (for the Colmez functor) and [Pa] (for equivalence of categories). We recall that Colmez has constructed a covariant exact functor $\mathbb{V}$

$$\mathbb{V} : \mathrm{Rep}_{\mathcal{O}}(\mathrm{GL}_2(\mathbb{Q}_p)) \to \mathrm{Rep}_{\mathcal{O}}(G_{\mathbb{Q}_p})$$

which sends $\mathcal{O}$-representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ to $\mathcal{O}$-representations of $G_{\mathbb{Q}_p} = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. Moreover this functor is compatible with deformations and induces an equivalence of categories when restricted to appropiate sub-representations. We call the inverse of this functor the p-adic local Langlands correspondence and we denote it by $B(\cdot)$. For our applications we will only need the fact that for p-adic continuous representations $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(K)$, $B(\rho)$ is a Banach admissible K-representation. Furthermore, when ρ is irreducible, then $B(\rho)$ is topologically irreducible.

Let $\bar{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(k)$ be a reduction of ρ. Let π be a supersingular representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over k which corresponds to $\bar{\rho}$ by the mod p local Langlands correspondence, that is $\mathbb{V}(\pi) = \bar{\rho}$. Then one knows that $B(\rho)$ is an object of the category $\mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p)}(K)_{(\pi)}$ defined above.

4.2. **Étale cohomology.** We recall results of Emerton on the p-adic completed cohomology and then we prove that $B(\rho_p)$ appears in the étale cohomology of $\mathcal{M}_{LT,\infty}$. From now on we work in the global setting. Let $\rho : G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(K)$ be a continuous Galois representation. We assume that it is unramified outside some finite set $\Sigma = \Sigma_0 \cup \{p\}$. Moreover we assume that its reduction $\bar{\rho}$ is modular (that is, isomorphic to a reduction of a Galois representation associated to some automorphic representation on $\mathrm{GL}_2(\mathbb{Q})$) and $\bar{\rho}_p = \bar{\rho}_{|G_{\mathbb{Q}_p}}$ is absolutely irreducible.

Let us define an adic space

$$X_\Sigma \sim \varprojlim_{K^p} \varprojlim_{K_p} X(K^p X_p)$$

where K_p runs over compact open subgroups of $\mathrm{GL}_2(\mathbb{Q}_p)$ and K^p runs over compact open subgroups of $\mathrm{GL}_2(\mathbb{A}_f^p)$ which are unramified outside Σ. Similarly we can define $X_{\Sigma,ss}, X_{\Sigma,ord}, Y_\Sigma, Y_{\Sigma,ss}, Y_{\Sigma,ord}$. We introduce those spaces to be able to talk conveniently about the action of the Hecke algebra $\mathbb{T}_\Sigma$ which we define below.

Let $\mathbb{T}_\Sigma = \mathcal{O}[T_l, S_l]_{l \notin \Sigma}$ be the commutative $\mathcal{O}$-algebra with T_l, S_l formal variables indexed by $l \notin \Sigma$. This is a standard Hecke algebra which acts on modular curves by correspondences.

Namely, T_l acts on Y_Σ and X_Σ (hence also on their cohomology) via double cosets

$$\mathrm{GL}_2(\mathbb{Z}_l)\begin{pmatrix}1 & 0\\ 0 & l\end{pmatrix}\mathrm{GL}_2(\mathbb{Z}_l)$$

and S_l via

$$\mathrm{GL}_2(\mathbb{Z}_l)\begin{pmatrix}l & 0\\ 0 & l\end{pmatrix}\mathrm{GL}_2(\mathbb{Z}_l)$$

To the modular Galois representation $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(K)$ we can associate the maximal Hecke ideal $\mathfrak{m}$ of $\mathbb{T}_\Sigma$ which is generated by ϖ (uniformiser of $\mathcal{O}$) and elements $T_l + a_l$ and $lS_l - b_l$, where l is a place of $\mathbb{Q}$ which does not belong to Σ, $X^2 + \bar{a}_l X^1 + \bar{b}_l$ is the characteristic polynomial of $\bar{\rho}(\mathrm{Frob}_l)$ and a_l, b_l are any lifts of $\bar{a}_l, \bar{b}_l$ to $\mathcal{O}$.

We let $\pi_{\Sigma_0}(\rho) = \otimes_{l\in\Sigma_0}\pi_l(\rho_l)$ be the tensor product of K-representations of $\mathrm{GL}_2(\mathbb{Q}_l)$ ($l \in \Sigma_0$) associated to $\rho_l = \rho_{|G_{\mathbb{Q}_l}}$ by the classical (modified) l-adic local Langlands correspondence (see [EH]).

We assume that ρ is pro-modular in the sense of Emerton (see [Em2]) and hence one can associate to it a prime ideal $\mathfrak{p}$ of $\mathbb{T}_\Sigma$ (similarly as we have associated $\mathfrak{m}$ to $\bar{\rho}$). Moreover, we have an obvious inclusion $\mathfrak{p} \subset \mathfrak{m}$. We remark that pro-modularity is a weaker condition than modularity and it can be seen as saying that ρ is a Galois representation associated to some p-adic automorphic representation of $\mathrm{GL}_2(\mathbb{Q})$. Recall that we have assumed that $\bar{\rho}_p = \bar{\rho}_{|G_{\mathbb{Q}_p}}$ is absolutely irreducible. This permits us to state the main result of [Em2] as

Theorem 4.1. *Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(K)$ be a continuous Galois representation which is pro-modular and such that $\bar{\rho}_p$ is absolutely irreducible. Then we have a $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{Q}_p) \times \prod_{l\in\Sigma_0}\mathrm{GL}_2(\mathbb{Q}_l)$-equivariant isomorphism of Banach admissible K-representations.*

$$H^1(Y_\Sigma, K)[\mathfrak{p}] \simeq \rho \otimes_K B(\rho_p) \otimes_K \pi_{\Sigma_0}(\rho)$$

We recall that the cohomology group on the left is the p-adic completed cohomology of Emerton

$$H^1(Y_\Sigma, K) = \left(\varinjlim_{K^p}\varprojlim_{s}\varinjlim_{K_p} H^1_{et}(Y(K^pK_p), \mathbb{Z}/p^s\mathbb{Z})\right)\otimes_{\mathbb{Z}_p} K$$

where K_p runs over compact open subgroups of $\mathrm{GL}_2(\mathbb{Q}_p)$ and K^p runs over compact open subgroups of $\mathrm{GL}_2(\mathbb{A}_f^p)$ which are unramified outside Σ.

Let us remark that the Galois action of $G_{\mathbb{Q}_p}$ arises on Y_Σ, X_Σ, $\mathcal{M}_{LT,\infty}$ (which we treat as adic spaces over $\mathrm{Spa}(C, \mathcal{O}_C)$) from the Galois action on a corresponding model over $\bar{\mathbb{Q}}_p$.

We also have a similar theorem for the compactification

Theorem 4.2. *With assumptions as in the theorem above, we have an isomorphism of Banach admissible K-representations*

$$H^1(X_\Sigma, K)_{\mathfrak{m}} \simeq H^1(Y_\Sigma, K)_{\mathfrak{m}}$$

In particular,

$$H^1(X_\Sigma, K)[\mathfrak{p}] \simeq \rho \otimes_K B(\rho_p) \otimes_K \pi_{\Sigma_0}(\rho)$$

Proof. We have assumed that $\bar{\rho}_p$ is absolutely irreducible and hence $\bar{\rho}$ is absolutely irreducible which implies that $\mathfrak{m}$ is a non-Eisenstein ideal. Now the theorem follows as in the proof of Proposition 7.7.13 of [Em3]. □

We now come back to the exact sequence which we have obtained earlier

$$... \to H^0(X_{ss}, K) \to H^1_{X_{ord}}(X, K) \to H^1(X, K) \to H^1(X_{ss}, K) \to ...$$

By Theorem 2.1.5 of [Em1], we get that $H^1(X, K)$ is a Banach admissible K-representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. Moreover, also $H^0(X_{ss}, K)$ is a Banach admissible K-representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ as $X_{ss}(K_p K^p)$ has only finite number of connected components for each K_p and K^p. The category of Banach admissible K-representations is closed under taking extensions of ϖ-adically continuous K-representations (see Proposition 2.4.11 of [Em4]). Hence, as $H^1_{X_{ord}}(X, K)$ is ϖ-adically continuous, we infer that it is also Banach admissible. By Proposition 2.4 we get that $H^1_{X_{ord}}(X, K)$ is induced from some representation $W(K)$ of the Borel $B(\mathbb{Q}_p)$. We deduce from Lemma 3.8 (Theorem 4.4.6 in [Em4]) that $W(K)$ is a Banach admissible K-representation of $B(\mathbb{Q}_p)$. Thus, we can apply to it Lemma 3.7. If π is any supersingular k-representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, it implies that

$$H^1_{X_{ord}}(X, K)_{(\pi)} = 0$$

because by Lemma 3.7 the only representations which appear in $H^1_{X_{ord}}(X, K)$ are of the form $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi_1 \otimes \chi_2)$, where χ_1, χ_2 are some K-characters of $\mathbb{Q}_p$.

Localising the exact sequence above at some supersingular k-representation π we get an injection

$$H^1(X, K)_{(\pi)} \hookrightarrow H^1(X_{ss}, K)$$

In the same vein we get an injection

$$H^1(X_\Sigma, K)_{(\pi)} \hookrightarrow H^1(X_{\Sigma,ss}, K)$$

for any finite set Σ as above. We can now prove our main theorem

Theorem 4.3. *Let $\rho : G_\mathbb{Q} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(K)$ be a pro-modular representation. Assume that $\bar{\rho}_p = \bar{\rho}_{|G_{\mathbb{Q}_p}}$ is absolutely irreducible. Then we have a $\mathrm{GL}_2(\mathbb{Q}_p) \times G_{\mathbb{Q}_p}$-equivariant injection*

$$B(\rho_p) \otimes_K \rho_p \hookrightarrow H^1(\mathcal{M}_{LT,\infty}, K)$$

Proof. Let π be the mod p representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ corresponding to $\bar{\rho}_p$ by the mod p local Langlands correspondence. It is a supersingular representation by our assumption that $\bar{\rho}_p$ is absolutely irreducible. Let $\mathfrak{p}$ be a prime ideal of $\mathbb{T}_\Sigma$ associated to ρ, where $\Sigma = \Sigma_0 \cup \{p\}$ is some finite set which contains p and all the primes at which ρ is ramified. As above we have

$$H^1(X_\Sigma, K)_{(\pi)} \hookrightarrow H^1(X_{\Sigma,ss}, K)$$

and hence also

$$H^1(X_\Sigma, K)_{(\pi)}[\mathfrak{p}] \hookrightarrow H^1(X_{\Sigma,ss}, K)[\mathfrak{p}]$$

Theorem 4.2 implies that (we keep track only of $G_{\mathbb{Q}_p}$-action instead of $G_\mathbb{Q}$)

$$B(\rho_p) \otimes_K \rho_p \otimes_K \pi_{\Sigma_0}(\rho) \hookrightarrow H^1(X_{\Sigma,ss}, K)[\mathfrak{p}]$$

Let K_{Σ_0} be a compact open subgroup of $\prod_{l\in\Sigma_0}\mathrm{GL}_2(\mathbb{Q}_l)$ for which we have $\dim \pi_{\Sigma_0}(\rho)^{K_{\Sigma_0}}=1$ where the dimension is over $\prod_{l\in\Sigma_0}\mathbb{Q}_l$. Such a subgroup always exists by classical results of Casselman (see [Cas]). Hence we have

$$B(\rho_p)\otimes_K \rho_p \hookrightarrow H^1(X_{\Sigma,ss},K)[\mathfrak{p}]^{K_{\Sigma_0}}$$

By Kunneth formula and Proposition 2.6 (the p-adic uniformisation of Rapoport-Zink) we get that

$$H^1(X_{ss},K)=\left(H^1(\mathcal{M}_{LT,\infty},K)\widehat{\otimes}_K\mathcal{S}\right)^{D^\times(\mathbb{Q}_p)}$$

where we have denoted by $\mathcal{S}$ the p-adic quaternionic forms

$$\widehat{H}^0(D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}),K)=\left(\varinjlim_{K^p}\varprojlim_{s}\varinjlim_{K_p} H^0(D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A})/K_pK^p,\mathbb{Z}/p^s\mathbb{Z})\right)\otimes_{\mathbb{Z}_p}K$$

where K_p and K^p run over compact open subgroups of $D^\times(\mathbb{Q}_p)$ and $D^\times(\mathbb{A}_f^p)$ respectively. For a finite set Σ we also have

$$H^1(X_{\Sigma,ss},K)=\left(H^1(\mathcal{M}_{LT,\infty},K)\widehat{\otimes}_K\mathcal{S}_\Sigma\right)^{D^\times(\mathbb{Q}_p)}$$

where $\mathcal{S}_\Sigma$ is defined similarly to $\mathcal{S}$ but with the limit running over K^p which are unramified outside Σ. As $\mathrm{GL}_2(\mathbb{Q}_p)$ and $G_{\mathbb{Q}_p}$ acts on $H^1(X_{\Sigma,ss},K)$ through $H^1(\mathcal{M}_{LT,\infty},K)$ we conclude by preceding discussion that

$$B(\rho_p)\otimes_K\rho_p\hookrightarrow H^1(\mathcal{M}_{LT,\infty},K)$$

as wanted. □

4.3. **Cohomology with compact support.** We show that the cohomology with compact support of the Lubin-Tate tower does not contain any p-adic representations which reduce to mod p supersingular representations. Recall we have morphisms

$$j:X_{ss}\hookrightarrow X$$

and

$$i:X_{ord}\to X$$

which give an exact sequence for any étale sheaf F on X

$$0\to j_!j^*F\to F\to i_*i^*F\to 0$$

This leads to an exact sequence of the p-adic completed cohomology

$$...\to H^0(X_{ord},K)\to H^1_c(X_{ss},K)\to H^1(X,K)\to H^1(X_{ord},K)\to ...$$

Because $H^1(X,K)$ is admissible as a K-representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ (by the result of Emerton) and $H^0(X_{ord},K)$ is admissible as a K-representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ because at each finite level X_{ord} has a finite number of connected components, we infer that also $H^1_c(X_{ss},K)$ is admissible (as the category of admissible K-representations is a Serre subcategory of ϖ-continuous K-representations). This means that we can localise $H^1_c(X_{ss},K)$ at supersingular representations.

Let π be a supersingular k-representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, where k is the residue field of K. Observe that if $H^1_c(X_{ss},K)_{(\pi)}\neq 0$, then also its reduction $H^1_c(X_{ss},k)_{(\pi)}$ would be non-zero. But Theorem 8.2 in [Cho] states that $H^1_c(X_{ss},k)_{(\pi)}=0$. Hence we get

Theorem 4.4. *For any supersingular k-representation π of $\mathrm{GL}_2(\mathbb{Q}_p)$ we have*

$$H^1_c(X_{ss}, K)_{(\pi)} = 0$$

In particular

$$H^1_c(\mathcal{M}_{LT,\infty}, K)_{(\pi)} = 0$$

Proof. The first part follows from the preceding discussion, the second part follows from the Rapoport-Zink uniformisation. □

This theorem implies that for any continuous $\rho_p : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(K)$ which has an absolutely irreducible reduction $\bar{\rho}_p : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{Q}_p)$, the $\mathrm{GL}_2(\mathbb{Q}_p)$-representation $B(\rho_p)$ associated to ρ_p by the p-adic Local Langlands correspondence does not appear in $H^1_c(\mathcal{M}_{LT,\infty}, K)$. Nevertheless, we do not know whether it can appear in $H^2_c(\mathcal{M}_{LT,\infty}, K)$ or not.

4.4. **Analytic cohomology.** If Z is any adic space, we denote by Z_{an} its analytic topos which arises from the topology of open subsets. For any (coherent) sheaf $\mathcal{F}$ on Z, we write $H^i_{an}(Z, \mathcal{F})$ for the i-th cohomology group of Z_{an} with values in $\mathcal{F}$.

We let $\mathcal{I} \subset \mathcal{O}_{X_\Sigma}$ be the ideal sheaf of the boundary of X_Σ. By Theorem IV.2.1 of [Sch2] (where we pass to the limit with $\mathbb{Z}/p^n\mathbb{Z}$ and K^p) we have an isomorphism

$$H^1(X_\Sigma, K)\widehat{\otimes}_K C \simeq H^1_{an}(X_\Sigma, \mathcal{I})$$

which is $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant and also equivariant with respect to the Hecke action of $\mathbb{T}_\Sigma$. Once again we look at the exact sequence, this time at the analytic level:

$$... \to H^0_{an}(X_{\Sigma,ss}, \mathcal{O}_{X_{\Sigma,ss}}) \to H^1_{an,X_{\Sigma,ord}}(X, \mathcal{I}) \to H^1_{an}(X_\Sigma, \mathcal{I}) \to H^1_{an}(X_{\Sigma,ss}, \mathcal{O}_{X_{\Sigma,ss}}) \to ...$$

where we have used the fact that $\mathcal{I}_{|X_{\Sigma,ss}}$ is equal to the structure sheaf $\mathcal{O}_{X_{\Sigma,ss}}$ because $X_{\Sigma,ss}$ does not meet the boundary.

If we were to use the same reasoning as for the p-adic completed cohomology to show that the p-adic local Langlands correspondence appears in the analytic cohomology of the Lubin-Tate tower at infinity, then we would have to start by proving admissibility of the cohomology groups. Unfortunately, this is not true. By the comparison theorem of Scholze we get that $H^1_{an}(X_\Sigma, \mathcal{I})$ is a Banach admissible K-represention, but $H^0_{an}(X_{\Sigma,ss}, \mathcal{O}_{X_{\Sigma,ss}})$ is not admissible (and it is not even clear whether it is a Banach space). In order to prove that, it is enough to prove it for $H^0_{an}(\mathcal{M}_{LT,\infty}, \mathcal{O}_{\mathcal{M}_{LT,\infty}})$ by the p-adic uniformisation of Rapoport-Zink.

Proposition 4.5. *The $GL_2(\mathbb{Q}_p)$-representation $H^0_{an}(\mathcal{M}_{LT,\infty}, \mathcal{O}_{\mathcal{M}_{LT,\infty}})$ is not admissible.*

Proof. In Section 2 (see especially 2.10) in [We], Weinstein gives an explicit description of the geometrically connected components of $\mathcal{M}_{LT,\infty}$. Each of them is isomorphic to $\mathrm{Spa}(A \otimes_{\mathcal{O}_{K_\infty}} C, A \otimes_{\mathcal{O}_{K_\infty}} \mathcal{O}_C)$, where K_∞ is the Lubin-Tate extension of $\mathbb{Q}_p$ (see Section 2.3 of [We]; we fix an embedding $K_\infty \hookrightarrow C$) and A is a perfectoid K_∞-algebra with a tilt (Corollary 2.9.11 of [We])

$$A^\flat \simeq \bar{\mathbb{F}}_p[[X_1^{1/p^\infty}, X_2^{1/p^\infty}]]$$

Hence, in $H^0_{an}(\mathcal{M}_{LT,\infty}, \mathcal{O}_{\mathcal{M}_{LT,\infty}})$ appears $A \otimes_{\mathcal{O}_{K_\infty}} C$ (and in fact much more as this is the set of all unbouded funtions on the Lubin-Tate perfectoid). We have an action of $\mathrm{GL}_2(\mathbb{Z}_p)$ on A. Let K be any compact open subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$. If $H^0_{an}(\mathcal{M}_{LT,\infty}, \mathcal{O}_{\mathcal{M}_{LT,\infty}})$ were admissible, then in particular for a lattice $A \otimes_{\mathcal{O}_{K_\infty}} \mathcal{O}_C$ in $A \otimes_{\mathcal{O}_{K_\infty}} C$, the reduction of K-invariants

$(A \otimes_{\mathcal{O}_{K_\infty}} \bar{\mathbb{F}}_p)^K$ would be of finite dimension over $\bar{\mathbb{F}}_p$ (by the very definition, see Definition 2.7.1 of [Em4]). This is not possible. Indeed, observe that A^K contains (and probably equals to but we do not need it) the ring of integral analytic functions on the Lubin-Tate space of K-level, which is a finite ring over the ring $\mathcal{O}_C[[X_1, X_2]]$ of power-series over $\mathcal{O}_C$. □

This means that we cannot use the localisation functor and deduce our result from the global results of Emerton. Hence, for now, we can only state a conjecture, which we believe to be a correct version of the folklore conjecture.

Conjecture 4.6. *Let $\rho_p : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$ be a continuous de Rham Galois representation. Then, there is a non-zero $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant injection*

$$B(\rho_p) \hookrightarrow H^1_{an}(\mathcal{M}_{LT,\infty}, \mathcal{O}_{\mathcal{M}_{LT,\infty}})$$

Observe that in fact we can state a similar conjecture for $H^0_{an}(\mathcal{M}_{LT,\infty}, \mathcal{O}_{\mathcal{M}_{LT,\infty}})$ instead of H^1_{an}. A priori, it is not clear which one should be true or whether both are. The advantage of working with H^0_{an} should be the fact that it is quite explicit by the work of Weinstein.

We believe that there is also a more refined version of the folklore conjecture which truly realizes the p-adic local Langlands correspondence in the sense that in the analytic cohomology of the Lubin-Tate perfectoid should appear a tensor product of $B(\rho_p)$ with the associated (ϕ, Γ)-module of ρ_p. We do not make precise here what kind of (ϕ, Γ)-modules we consider and how the appropiate Robba ring acts on the Lubin-Tate perfectoid. We shall come back to those issues elsewhere.

4.5. **Final remarks.** Observe that our proof of Theorem 4.3 depends on the global data as we have to start with a global pro-modular Galois representation ρ. As our result is completely local, it is natural to ask whether the same thing holds for any absolutely irreducible Galois representation ρ_p of $G_{\mathbb{Q}_p}$ which is not neccessarily a restriction of some global ρ (as in Conjecture 4.6).

Another natural problem is to try to prove Theorem 4.3 without assuming that $\bar{\rho}_p$ is absolutely irreducible. This would require a more careful study of the cohomology of the ordinary locus.

The most pertaining problem is whether one can reconstruct $B(\rho_p)$ from either the p-adic completed or the analytic cohomology of the Lubin-Tate tower and hence give a different proof of the p-adic local Langlands correspondence. This might be crucial in trying to prove the existence of the p-adic correspondence for groups other than $\mathrm{GL}_2(\mathbb{Q}_p)$ as well as Theorem 4.3 for Galois representations ρ_p not neccessarily coming from global Galois representations.

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