DRINFELD LEVEL STRUCTURES AND LUBIN-TATE SPACES

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Le but principal de cet exposé est de demontrer la regularité des anneaux representants les foncteurs des deformations des groupes formels avec la structure de niveau de Drinfeld. Je suis la demonstration dans section 4 de [**Dr**]. Pour les groupes p-divisibles, je suis II.2 dans [**HT**]. Pour la demonstration des propriétés étales et indication des actions des groups, je suis [**Str**].

Remark 0.1. — Pour la compatibilité avec l'exposé de Gabriel Dospinescu, j'ai decidé d'écrire ces notes en anglais. Ils feront la partie de groupe de travail sur la preuve de Peter Scholze de la correpondence locale de Langlands ([**Sch**]).

1. Reminder

Let K be a finite extension of $\mathbb{Q}_p$, $\mathcal{O} = \mathcal{O}_K$ its ring of integers, ϖ uniformiser, $k = \mathcal{O}/\varpi$ residue field, q the number of elements in k. Let $\widehat{\mathcal{O}}^{nr}$ be the completion of the maximal unramified extension of $\mathcal{O}$. I will briefly recall what we have seen in last few talks. See also [**Dr**] for the proofs.

Definition 1.1. — Let S be a schema. A p-divisible group G/S is a sheaf on $(Sch/S)_{fppf}$ such that $G = \varinjlim_n G[p^n]$, each $G[p^n]$ is finite, locally free on S and $p : G \to G$ is surjective.

A ϖ-divisible $\mathcal{O}$-module G/S is a p-divisible group G over an $\mathcal{O}$-scheme S with an action $\mathcal{O} \to End_{\mathcal{O}_S}(G)$ which is compatible with an action of $Lie(G)$.

A height of a p-divisible group G/S is an integer h such that $|G[p]| = q^h$.

We have analogous definitions on the "'formal side"':

Definition 1.2. — Let R be a ring. A formal group F/R of dimension n are series $\underline{F}(\underline{X}, \underline{Y}) \in R[[X_1, ..., X_n, Y_1, ..., Y_n]]$ which satisfy "'group axioms"' (i.e. are group objects in the appropiate category of formal schemes).

A formal $\mathcal{O}$-module F/R is a formal group F over an $\mathcal{O}$-algebra R with a morphism $\theta : \mathcal{O} \to End_R(F)$ compatible with the natural homomorphism $\mathcal{O} \to R$.

A height of a F/R (for R in which $\varpi = 0$) is h such that $\varpi_F(X) = f(x^{q^h})$ and $f'(0) \neq 0$, where ϖ_F denotes series of multiplication by ϖ_F on F.

There is a following connection between these two notions. Let G be a p-divisible group over R. There is an exact sequence of p-divisible groups:

$$0 \to G^0 \to G \to G^{ét} \to 0$$

where G^0 is infinitesimal and $G^{ét}$ is étale. We have

$$\{\text{infinitesimal p-divisible groups of dim } n/R\} \simeq \{\text{formal groups of dim } n/R\}$$

Remark 1.3. — We also have

$$\{\text{étale p-divisible groups}/R\} \simeq \{\text{finite, torsion-free, smooth, étale } \mathcal{O} - \text{sheaves on } /R\}$$

Assumption: From now on, we will assume that considered formal and p-divisible groups have dimension 1.

In the last talks, we have seen proofs of

Proposition 1.4. — *All formal $\mathcal{O}$-modules of height $h < \infty$ over a separably closed field k are isomorphic (say, to a $\Sigma_{k,h}$).*

Corollary 1.5. — *All ϖ-divisible $\mathcal{O}$-modules over k are of the form $\Sigma_{k,h} \times (K/\mathcal{O})^g$ for certain integers g and h.*

Let C be the category of complete, local, noetherian $\widehat{\mathcal{O}}^{nr}$-algebras.

Proposition 1.6. — *Let F be a formal O-module over k of height $h < \infty$. The functor:*

$$R \in C \mapsto \{\text{deformations of } F \text{ over } R\}/\simeq$$

is representable by $\widehat{\mathcal{O}}^{nr}[[t_1, ..., t_{h-1}]]$.

2. Drinfeld level structures

For $R \in C$ and a formal group F/R we will define height by $ht(F) = ht(F \otimes_R k)$.

Definition 2.1. — A Drinfeld structure of level n on a formal $\mathcal{O}$-module F/R of height h is a homomorphism

$$\phi : (\varpi^{-n}\mathcal{O}/\mathcal{O})^h \to F[\varpi^n](R)$$

such that $\prod_{x \in (\varpi^{-1}\mathcal{O}/\mathcal{O})^h}(T - \phi(x)) | \varpi_F(T)$.

Proposition 2.2. — *For $R = \widehat{\mathcal{O}}^{nr}/\varpi$ there exists a unique structure of level n on F/R, that is, trivial level structure: $\phi^{triv} : (\varpi^{-n}\mathcal{O}/\mathcal{O})^h \to F[\varpi^n](R)$ given by $x \mapsto 0$ for all x.*

Proof. — As R is reduced and $F[\varpi^n]$ is radicial over R, we have $F[\varpi^n](R) = \{0\}$, hence there exists at most one Drinfeld level structure. It rests to check that ϕ^{triv} is indeed a level structure. ☐

Remark 2.3. — A morphism $f : X \to Y$ between schemes is radicial if for each field K, the map $X(K) \to Y(K)$ is injective.

Remark 2.4. — The proof works for any reduced $R/\mathbb{F}_p$. We will recall this fact later. See also II.2.1.3 in [**HT**].

3. Deformations

By a deformation of level n, we will mean a deformation with a Drinfeld structure of level n. Fix a formal $\mathcal{O}$-module G over k. We will prove the following theorem of Drinfeld (see proposition 4.3 in [**Dr**]).

Theorem 3.1. — *1) Functor*

$$R \in C \mapsto \{(X, \iota, \phi) | X \text{ is a formal } \mathcal{O}\text{-module over } R,\ \iota : F \simeq X_k, \phi \text{ is an } n\text{-level structure on } X\}/\simeq$$

is representable by a ring R_n.

2) R_n is regular. Let $n \geq 1$, $e_i (i = 1, ..., h)$ be a base of $(\varpi^{-n}\mathcal{O}/\mathcal{O})^h$ as $\mathcal{O}/\varpi^n$-module. The images of e_i in R_n under the universal deformation of level n form a system of local parameters for R_n.

3) For $n \geq m$, the homomorphism $R_m \to R_n$ is finite and flat.

Proof. — Let F be a universal deformation over $R_0 \simeq \widehat{\mathcal{O}}^{nr}[[t_1, ..., t_{h-1}]]$. For $0 \leq r \leq h$, consider the functor Φ_r which associates to each R_0-algebra $R \in C$, the set of homomorphisms $\phi : (\varpi^{-1}\mathcal{O}/\mathcal{O})^r \to F[\varpi](R)$ such that $\prod_{x \in (\varpi^{-1}\mathcal{O}/\mathcal{O})^r}(T - \phi(x)) | \varpi_F(T)$. Let us prove:

Lemma 3.2. — *The functor Φ_r is represented by a ring L_r having the following properties:*

1) L_r is regular. Let $e_i (i = 1, ..., r)$ be a base of $(\varpi^{-1}\mathcal{O}/\mathcal{O})^r$. The images of e_i in L_r under the universal deformation of level n and $t_{r+1}, ...t_{h-1}$ form a system of local parameters for L_r.

2) The homomorphism $L_{r-1} \to L_r$ is finite and flat.

Proof. — We prove the lemma by induction. For $r=0$ it is true. Suppose it is proves for $r-1$ and let $\phi_{r-1} : (\varpi^{-1}\mathcal{O}/\mathcal{O})^r \to F[\varpi](R)$ be the universal structure on L_{r-1}. Put $\theta_i = \phi_{r-1}(e_i)(i=1,...,r-1)$ and

$$g(T) = \frac{\varpi_{L_{r-1}}(T)}{\prod_{x\in(\varpi^{-1}\mathcal{O}/\mathcal{O})^r}(T-\phi_{r-1}(x))}$$

Let $L_r = L_{r-1}[[\theta_r]]/g(\theta_r)$. We define a homomorphism $\phi_r : \varpi^{-1}\mathcal{O}/\mathcal{O})^{r-1} \oplus \varpi^{-1}\mathcal{O}/\mathcal{O}) \to L_r$ by taking $(\phi_r)_{|\varpi^{-1}\mathcal{O}/\mathcal{O})^{r-1}} = \phi_{r-1}$ and $(\phi_r)_{|\varpi^{-1}\mathcal{O}/\mathcal{O})}$ sends ϖ^{-1} to θ_r. By using Weierstrass factorisation theorem for a serie $\varpi_{L_{r-1}}$, we see that L_r is finite and flat over L_{r-1} and moreover $L_{r-1}/(\theta_1,...,\theta_r,t_{r+1},...,t_{h-1}) = \widehat{\mathcal{O}}^{nr}/\varpi$ and so L_r is regular and $\theta_1,...,\theta_r,t_{r+1},...,t_{h-1}$ form a system of local parameters for L_r. It suffices to see that L_r represents Φ_r and for that, we should show that $\prod_{x\in(\varpi^{-1}\mathcal{O}/\mathcal{O})^r}(T-\phi(x))|\varpi_{L_r}(T)$. But, by induction and definitions of θ_r and $g(T)$, the serie $\varpi_{L_r}(T)$ is divisible by each $T-\phi_r(x)$ for $x\in(\varpi^{-1}\mathcal{O}/\mathcal{O})^r$. As L_r is regular, hence integral, it suffices to show that ϕ_r is injective. But if $\phi_r(\sum_{i=1}^{l}\alpha_i e_i)=0$, then $\sum_{i=1}^{r}\alpha_i\theta_i = 0$ in $F[\varpi](R)$ hence $\alpha_i\in(\varpi)$. This shows injectivity.

□

Setting in the lemma $r=h$, we obtain the theorem for $n=1$, i.e. $L_h = R_1$. Now, we also use induction starting at 1. Suppose $n\geq 1$ and that theorem has been showed for R_n. Let $e_i(i=1,...,h)$ be a base of $(\varpi^{-n}\mathcal{O}/\mathcal{O})^h$ and let b_i be the image of e_i in R_n. We have, by definition of level structure,

$$R_{n+1} = R_n[[y_1,...,y_h]]/(\varpi_{R_n}(y_1)-b_1,...,\varpi_{R_n}(y_h)-b_h)$$

hence, R_{n+1} is regular and $(y_1,...,y_h)$ is a system of local parameters for R_{n+1} and also the homomorphism $R_n\to R_{n+1}$ is finite and flat. □

Let $K_0 = GL_n(\mathcal{O})$ and let $K_m = 1+\varpi^m M_n(\mathcal{O})$ for $m\geq 1$.

***Proposition 3.3* ([Str], 2.1.2).** — *For $n\geq m$, $R_n[\frac{1}{\varpi}]$ is étale and Galois over $R_m[\frac{1}{\varpi}]$ with a Galois group isomorphic to K_m/K_n.*

Sketch of a proof. — We can suppose $m=0$. Firstly, one takes $n=1$ and hence $R_1 = L_h$ in the notation of the above proposition. By Weierstrass factorisation $\varpi_{R_0}(T) = P(T)e(T)$ where P is a polynomial in $R_0[T]$ and e is an inversible series in $R_0[[T]]$. We prove that $P(T)$ is separable over $R_0[\frac{1}{\varpi}]$ (see 2.1.2 in **[Str]** for explicit computations) and hence $L_i[\frac{1}{\varpi}]/L_{i-1}[\frac{1}{\varpi}]$ are étale for each i. By the construction of R_1 (see the proof of the above proposition), this implies that $R_1[\frac{1}{\varpi}]/R_0[\frac{1}{\varpi}]$ is étale.

For $n>1$, recall the description $R_n = R_{n-1}[[y_1,...,y_h]]/(\varpi_{R_{n-1}}(y_1)-b_1,...,\varpi_{R_{n-1}}(y_h)-b_h)$. By Weierstrass factorisation we have $\varpi_{R_{n-1}}(T)-b_i = P_{n,i}(T)e_{n,i}(T)$ where $P_{n,i}$ is a polynomial and $e_{n,i}$ an inversible series and we show that $P_{n,i}$ has simple zeroes only outside the vanishing locus of ϖ, by explicit computations of derivative $\varpi_{R_{n-1}}(T)'$. This shows that $R_n[\frac{1}{\varpi}]/R_0[\frac{1}{\varpi}]$ is étale.

To prove that it is Galois, use the fact that the universal level structure is injective (it was proved for R_1 in the proof of the lemma above).

□

4. p-divisible groups

We will give another definition of Drinfeld level structure for p-divisible groups.

Definition 4.1. — A structure of level n on a ϖ-divisible $\mathcal{O}$-module G of the constant height $h<\infty$ over $\mathcal{O}$-scheme S is a morphism of $\mathcal{O}$-modules

$$\phi : (\varpi^{-n}\mathcal{O}/\mathcal{O})^h \to G[\varpi^n](S)$$

such that $\phi(x)$ for $x\in(\varpi^{-n}\mathcal{O}/\mathcal{O})^h$ form full set of sections of $G[\varpi^n](S)$, i.e. for every affine S-scheme $Spec(R)$ and every $f\in H^0(G[\varpi^n]_R,\mathcal{O})$ we have an equality of polynomials in $R[T]$:

$$det(T-f) = \prod_{x\in(\varpi^{-n}\mathcal{O}/\mathcal{O})^h}(T-f(\phi(x)))$$

Remark 4.2. — The last condition is also equivalent to

$$Norm(f) = \prod_{x \in (\varpi^{-n}\mathcal{O}/\mathcal{O})^h} f(\phi(x))$$

Proposition 4.3 **([HT], II.2.3)**. — *If $R \in C$ and G is a one-dimensional, p-divisible infinitesimal group, i.e. $G \simeq Spf R[[T]]$, then the above definition of level structure and the definition we have given before for formal groups coincide.*

Idea of a proof. — We show that this second definition also gives us a functor representable by a ring R_n'. Then one works on the "‘universal level"’ to show that $R_n = R_n'$. □

Proposition 4.4 **([HT], II.2.1.4)**. — *Let S be connected, and let G/S be a ϖ-divisible $\mathcal{O}$-module. Consider the following diagram (which may not always exist):*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G^0[\varpi^n](S) & \longrightarrow & G[\varpi^n](S) & \longrightarrow & G^{\acute{e}t}[\varpi^n](S) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \phi_{|M}} & & \uparrow{\scriptstyle \phi} & & \uparrow{\scriptstyle \phi^M} & & \\ 0 & \longrightarrow & M & \longrightarrow & (\varpi^{-n}\mathcal{O}/\mathcal{O})^h & \longrightarrow & (\varpi^{-n}\mathcal{O}/\mathcal{O})^h/M & \longrightarrow & 0 \end{array}$$

Then to give a n-level structure ϕ is equivalent to giving a direct factor $M \subset (\varpi^{-n}\mathcal{O}/\mathcal{O})^h$ such that $\phi_{|M}$ is m-level structure and an isomorphism $\phi^M : (\varpi^{-n}\mathcal{O}/\mathcal{O})^h/M \simeq G^{\acute{e}t}[\varpi^n](S)$

Proof. — It results from the general fact on the extension of étale groups. See 1.11.2 in [**KM**]. □

Proposition 4.5 **([HT], II.2.1.5)**. — *Consider the same situation as in the proposition above, but assume moreover that S is reduced, connected and $p = 0$ in S. Then, if there exists a n-level structure then there exists a unique splitting $G[\varpi^n] \simeq G^0[\varpi^n] \times G^{\acute{e}t}[\varpi^n]$ on S. Also, if there exists a splitting $G[\varpi^n] \simeq G^0[\varpi^n] \times G^{\acute{e}t}[\varpi^n]$, then giving an n-level structure is the same as giving a direct factor $M \subset (\varpi^{-n}\mathcal{O}/\mathcal{O})^h$ and an isomorphism $\phi^M : (\varpi^{-n}\mathcal{O}/\mathcal{O})^h/M \simeq G^{\acute{e}t}[\varpi^n](S)$*

Sketch of a proof. — If ϕ is a level structure, then $M = ker\phi$ and by the proposition above, and the proposition 2.4, we have that $\phi^M : (\varpi^{-n}\mathcal{O}/\mathcal{O})^h/M \to G[\varpi^n](S) \to G^{\acute{e}t}[\varpi^n](S)$ is an isomorphism. The splitting is given by the image of $(\varpi^{-n}\mathcal{O}/\mathcal{O})^h/M$ in $G[\varpi^n](S)$. The proof of uniqueness is left to the reader. □

5. Group actions

Let $K_0 = GL_n(\mathcal{O})$ and let $K_m = 1 + \varpi^m M_n(\mathcal{O})$ for $m \geq 1$. Fix a formal $\mathcal{O}$-module F/k and let us define for $R \in C$ the functor $\mathcal{M}_{K_m}$ by:

$\mathcal{M}_{K_m}(R) = \{(X, \iota, \phi) | X$ is a formal $\mathcal{O}$-module over R , $\iota : F \simeq X_k, \phi$ is an m-level structure on X$\}/\simeq$

which we know to be representable by R_m.

Let $B = End_{\mathcal{O}}(F) \otimes_{\mathcal{O}} K$. It is a division algebra with invariant $\frac{1}{h}$. First of all observe that there is an action of B on $\mathcal{M}_{K_m}$ by

$$(X, \iota, \phi) \cdot b = (X, \iota \circ b, \phi)$$

Secondly, we show how $G = GL_n(K)$ acts on $(\mathcal{M}_{K_m})_m$. Let $g \in G$ and suppose that $g^{-1} \in M_n(\mathcal{O})$. For $n \geq m \geq 0$ such that $g\mathcal{O}^h \subset \varpi^{-(n-m)}\mathcal{O}^h$ (inclusion is considered in F^n), we will define a natural transformation $g_{n,m} : \mathcal{M}_{K_n} \to \mathcal{M}_{K_m}$ and for $(X, \iota, \phi) \in \mathcal{M}_{K_m}$ we will write

$$(X, \iota, \phi) \cdot g = g_{n,m}(X, \iota, \phi) = (X', \iota', \phi')$$

Let us define (X', ι', ϕ'). Condition on g implies that $\mathcal{O}^h \subset g\mathcal{O}^h$ so $(g\mathcal{O}/\mathcal{O})^h$ is a subgroup of $(\varpi^{-n}\mathcal{O}/\mathcal{O})^h$ and we can define

$$X' = X/\phi((g\mathcal{O}/\mathcal{O})^h)$$

which makes sense by proposition 4.4 in [**Dr**]. Moreover, a left multiplication by g induces an injective homomorphism

$$(\varpi^{-m}\mathcal{O}/\mathcal{O})^h \to (\varpi^{-n}\mathcal{O}/g\mathcal{O})^h = (\varpi^{-n}\mathcal{O}/\mathcal{O})^h/(g\mathcal{O}/\mathcal{O})^h$$

and the composition with the morphism induced by ϕ:

$$(\varpi^{-n}\mathcal{O}/\mathcal{O})^h/(g\mathcal{O}/\mathcal{O})^h \to X/\phi((g\mathcal{O}/\mathcal{O})^h) = X'$$

gives (again by proposition 4.4 of [**Dr**]) an m-level structure:

$$\phi' : (\varpi^{-m}\mathcal{O}/\mathcal{O})^h \to X'[\varpi^m](R)$$

Finally, define ι' to be the composition of ι with the projection $X_k \to X'_k$.

For arbitrary $g \in G$, take $r \in \mathbb{Z}$ such that $(\varpi^{-r}g)^{-1} \in M_n(\mathcal{O})$ and hence for $n \geq m \geq 0$ such that $\varpi^{-r}g\mathcal{O}^h \subset \varpi^{-(n-m)}\mathcal{O}^h$ if we define $(X', \iota', \phi') = (X, \iota, \phi) \cdot (\varpi^{-r}g)$ then define

$$(X, \iota\phi) \cdot g = (X', \iota' \circ \varpi^{-r}, \phi')$$

This gives a natural transformation $g_{n,m} : \mathcal{M}_{K_n} \to \mathcal{M}_{K_m}$ which does not depend on r nor on ϖ. In particular, for every m there is an action of $K_0 = GL_n(\mathcal{O})$ on $\mathcal{M}_{K_m}$ which commutes with an action of B.

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