E. Hellmann GDT "perfectoid spaces" - overview

Def. A perfectoid field is a non-archimedean field K complete w.r.t. $|\cdot|: K \to \mathbb{R}_{\geq 0}$ s.t. $|K| \subset \mathbb{R}$ dense and $\varphi = \mathrm{Frob}: \mathcal{O}_K/p \to \mathcal{O}_K/p$ is surjective.

Basic examples: $\cdot$ $K/\mathbb{Q}_p$ s.t. $K = \hat{\bar{K}}$ $\Rightarrow$ K perfectoid

$\cdot$ $\mathbb{Q}_p(\mu_{p^\infty})^\wedge$, $\mathbb{Q}_p(p^{1/p^\infty})^\wedge$, $\mathbb{F}_p((t^{1/p^\infty})) = \mathbb{F}_p((t))[t^{1/p^\infty}]^\wedge$.

in what follows: fix perfectoid base field K and $\varpi \in K$ s.t. $|p| \leq |\varpi| < 1$.

Tilting: $\mathcal{O}_K = K^\circ \subseteq K$ powerbounded elts.; $K^{\flat,\circ} := \varprojlim_{\mathrm{Frob}} K^\circ/\varpi$. Ring in char p.

$K^\flat = \mathrm{Frac}(K^{\flat,\circ})$ "tilt of K"

Remark (i) $K^\flat$ is a perfectoid field of char p.

(ii) $\varpi \in K^\circ \longleftrightarrow \varpi^\flat \in K^{\flat,\circ}$, "even better": $K^{\flat,\circ} \cong \varprojlim_{x \mapsto x^p} K^\circ$ (as mult. monoids) yields a map $K^\flat \to K$, $f \mapsto f^\#$

induced by $K^{\flat,\circ} = \varprojlim K^\circ \longrightarrow K^\circ$

projection to the 1st factor.

(iii) if char $K = 0$, consider the map

$$\theta: W(K^{\flat,\circ}) \twoheadrightarrow K^\circ \quad \text{which is a surjection.}$$
$$[x] \longmapsto x^\#$$

Prop.1 (fix K) Tilting induces an equivalence of categories

$$\{\text{fin. ext. of } K\} \overset{1-1}{\longleftrightarrow} \{\text{fin. ext. of } K^\flat\}$$
$$L \longmapsto L^\flat$$

(in case of char $K = 0$, inverse is given by $S \longmapsto W(S^\circ) \otimes_{W(K^{\flat,\circ}),\theta} K$)

especially: $\mathrm{Gal}(K^{sep}/K) \cong \mathrm{Gal}(K^{\flat,sep}/K^\flat)$

Prop.2 There is a bijection: $\{\text{cont. valuations on } K\} \overset{1-1}{\longleftrightarrow} \{\text{cont. valuations on } K^\flat\}$

s.t. $\forall_{x \in K^\flat}$ $|x|^\flat = |x^\#|$. $\qquad |\cdot| \longmapsto |\cdot|^\flat$

idea of proof: $\{\text{cont. val. on } K\} \overset{1-1}{\longleftrightarrow} \{\text{open and integrally closed subrings of } K^\circ\}$

Def. A perfectoid K-algebra is a Banach K-alg. R s.t. $R^\circ \subseteq R$ is bounded and $\phi = \mathrm{Frob}: R^\circ/p \twoheadrightarrow R^\circ/p$ is surjective.

Rem. (i) if char $K = p$, then $\Leftrightarrow$ R perfect.

(ii) R perfectoid $\Rightarrow$ R reduced (follows from boundedness of R°)

Examples: $R = K\langle T^{1/p^\infty}\rangle$

Tilting for algebras: given R as above, $R^{\flat,\circ} = \varprojlim_{x\mapsto x^p} R^\circ/\varpi$, $R^\flat = R^{\flat,\circ}\otimes_{K^{\flat,\circ}} K^\flat = R^{\flat,\circ}[\frac{1}{\varpi^\flat}]$

Def. (i) A perfectoid K°-alg. is a complete and flat $K^{\circ,a}$-alg. A s.t.
(integral level) $\phi: A/\varpi^{1/p} \xrightarrow{\cong} A/\varpi$ isom in cat. of $K^{\circ a}$-alg.

(ii) A perfectoid (K°/ϖ)-alg is —"— $(K^\circ/\varpi)^a$-alg. s.t. —"— .

Thm. (Tilting equivalence) There is a chain of equivalences of categories:

$$K\text{-Perf} \cong K^\circ\text{-Perf} \cong K^\circ/\varpi\text{-Perf} = K^{\flat,\circ}/\varpi^\flat\text{-Perf} \cong K^{\flat,\circ}\text{-Perf} \cong K^\flat\text{-Perf}.$$

$$R \longmapsto R^\flat .$$

Rem. if $\mathrm{char}\, K = 0$, inverse map is $S \mapsto W(S^\circ)\otimes_{W(K^{\flat,\circ}),\gamma} K$.

idea of proof: ad(i): M is a $K^{\circ,a}$-module then M is flat as $K^{\circ,a}$-mod if and only if $M_* = \mathrm{Hom}_{K^{\circ,a}}(K^{\circ,a}, M)$ is flat K°-mod ($\Leftrightarrow M_*$ has no ϖ-torsion)

M is ϖ-adically complete $\Leftrightarrow M_*$ is ϖ-adically complete.

$R \longmapsto R^{\circ,a}$ and $S_*[\frac{1}{\varpi}] \longleftarrow S$ are inverse to each other.

ad(ii) one needs to show: $A \longmapsto A/\varpi$ is an equivalence.

Want to show: deforming A/ϖ is unobstructed; most important step:

if $\bar A$ is perfectoid $K^{\circ,a}/\varpi$-alg. $\Rightarrow \mathbb{L}^a_{\bar A/(K^{\circ,a}/\varpi)} = 0$ ← cotangent complex (Illusie)

Assume A_n a lift of $\bar A$ to $K^{\circ,a}/\varpi^n$ is constructed.

$\Rightarrow$ exact triangle $\mathbb{L}^a_{\bar A/(K^\circ/\varpi)} \xrightarrow{\varpi^{n-1}} \mathbb{L}^a_{A_n} \longrightarrow \mathbb{L}^a_{A_{n-1}}$ $\Rightarrow$ by induction $\mathbb{L}^a_{A_n/(K^\circ/\varpi^n)} = 0$

$\Rightarrow$ no obstruction and no ambiguity to lifting.

Def. let R be perfectoid K-alg., $R^+ \subseteq R$ open and integral closed K°-alg. Then we say that $X = \mathrm{Spa}(R, R^+)$ is an affinoid perfectoid space.

Topology on X is induced by rational subsets $\mathcal{U}(\frac{f_1,\dots,f_n}{g}) = \{x \mid |f_i(x)| \le |g(x)|\}$: $f_i, g \in R$, $(f_i) = R$

X is equipped with presheaves $\mathcal{O}_X, \mathcal{O}_X^+$. Those are sheaves for perf. spaces.

Remark: the role of R^+: easy to see $\mathfrak{m} R^\circ \subseteq R^+ \subseteq R^\circ$ where $\mathfrak{m}$ is K° maximal.

Def. A perfectoid space is a locally ringed space $(X, \mathcal{O}_X)$ together with the system of continuous valuations $|\cdot|_x : \mathcal{O}_{X,x} \longrightarrow \Gamma \cup \{0\}$ $\forall x \in X$ s.t. locally $(X, \mathcal{O}_X, |\cdot|_x)$ is affinoid perfectoid.

Tilting: K perfectoid field, $K^\flat$ tilt, R perf. K alg, $R^\flat$ its tilt.
Then $X = \mathrm{Spa}(R, R^+) \mapsto X^\flat = \mathrm{Spa}(R^\flat, R^{\flat +})$ its tilt.

recall: we have a map $R^\flat \longrightarrow R$ $f \longmapsto f^\#$ (just like for fields)

Thm. There exists a homeomorphism: $h: X = \mathrm{Spa}(R, R^+) \xrightarrow{\cong} X^\flat = \mathrm{Spa}(R^\flat, R^{\flat +})$
$x \longmapsto x^\flat$.
s.t. $|f(x^\flat)| = |f^\#(x)|$ $\forall_{f \in R}$

and $U \subseteq X$ is a rational subset $\iff$ $h(U) \subseteq X^\flat$ is rational
$h(U) = U^\flat$

s.t. if $U^\flat = \{x \in X^\flat \mid |f_i(x)| \leq |g(x)|\}$
then $U = \{x \in X \mid |f_i^\#(x)| \leq |g^\#(x)|\}$

Moreover, $\Gamma(U^\flat, \mathcal{O}^+_{X^\flat})^a = R^{\flat,\circ}\langle\left(\frac{f_1,\dots,f_n}{g}\right)^{1/p^\infty}\rangle$ and also $\Gamma(U, \mathcal{O}_X^+)^a = R^\circ\langle\left(\frac{f_i^\#}{g}\right)^{1/p^\infty}\rangle$

especially: $\Gamma(U, \mathcal{O}_X)^\flat = \Gamma(U^\flat, \mathcal{O}_{X^\flat})$

About the proof: for any $f \in R$, $c \geq 0$, $\varepsilon > 0$ $\exists g_{c,\varepsilon} \in R^\flat$ s.t.
$$|f(x) - g_{c,\varepsilon}^\#| \leq |\varpi|^{1-\varepsilon} \max(|f(x)|, |\varpi|^c)$$
$\Longrightarrow$ each rational subset of X is a pre-image of a rational subset in $X^\flat$.
injectivity of h follows as X and $X^\flat$ are T_0-spaces.

Lemma: $x \in X^\flat$ $\to$ $\widehat{k(x)}$ = completion of res. field at x is a perfectoid field.
$\widehat{k(x)}^{\circ a} = \varpi^\flat$-adic completions of $\mathcal{O}_{X,x}^{\circ,a}$ $\to$ perfectoid $\Rightarrow$ $\widehat{k(x)}$ perfectoid.

Surjectivity: Let $x \in X^\flat$ $\Rightarrow$ $\widehat{k(x)}$ perfectoid field $/K^\flat$. can "undo" the tilt $\rightsquigarrow$
$\rightsquigarrow$ find field L/K s.t. $L^\flat = \widehat{k(x)}$ and $\{$valuations on $L\} \xleftrightarrow{1-1} \{$valuations on $\widehat{k(x)}\}$
$y \longmapsto x$
in a way s.t. $|f(x)| = |f^\#(y)|$. $\square$

Cor. $x \in X \Rightarrow \widehat{k(x)}$ perfectoid.

Acyclicity thm $X = \mathrm{Spa}(R, R^+)$ aff. perfectoid. Assume $X = \bigcup U_i$ fin. many rational subsets

(i) the sequence is exact $0 \to \Gamma(X, \mathcal{O}_X) \to \prod \Gamma(U_i, \mathcal{O}_X) \to \prod \Gamma(U_i \cap U_j, \mathcal{O}_X) \to \dots$

(ii) $0 \to \Gamma(X, \mathcal{O}_X^+) \to \prod \Gamma(U_i, \mathcal{O}_X^+) \to \dots$ is almost exact.

idea of proof: only prove (ii); first in char p; reduce to case R = perfection of K-alg. S top. of finite type
$\Rightarrow$ known that cohom. of $\underbrace{\text{complex in (ii)}}_{\text{for } S}$ is ϖ-power-torsion.
$\Rightarrow$ vanishes after perfection.

for char 0: tilt the statement for char p.

<u>Almost purity</u>: <u>def</u>: (i) A morphism $Y \to X$ of perf. spaces$/_k$ is <u>finite étale</u>

if $\forall U \subseteq X$ aff. perf. $U = \mathrm{Spa}(R, R^+)$ we have $f^{-1}(U) \subseteq Y$ aff. perf $= \mathrm{Spa}(S, S^+)$

s.t. $S/_R$ finite étale and $S^+ \subseteq S$ integral closure of R^+.

<u>étale</u>: "locally factors into immersion to something finite étale"

<u>Almost purity thm</u> (version of Scholze).

Tilting induces an equivalence

$$\left\{\begin{array}{c}\text{finite étale}\\ \text{covers of } X\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{finite étale}\\ \text{covers of } X^\flat\end{array}\right\}$$

$$Y \longmapsto Y^\flat$$

<u>remark</u>: "almost" refers to: $S/_R$ fin. ét. $\Leftrightarrow$ $S^+/_{R^+}$ almost fin. ét.

<u>idea of proof</u>: Let $x \in X \rightsquigarrow x^\flat \in X^\flat$, we find:

$$2\text{-}\varinjlim_{x \in U} U_{\text{fét}} \cong \widehat{k(x)}_{\text{fét}} \cong \widehat{k(x^\flat)}_{\text{fét}} \cong 2\text{-}\varinjlim_{x^\flat \in U^\flat} U^\flat_{\text{fét}}$$

we have to "glue" those isomorphisms. $\square$