GDT "p-div$_{O_C}$ II" (1)

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Aim: Thm (~~Scholze-Weinstein~~) Let $C$ alg closed $/\mathbb{Q}_p$, complete. Then there exists an equiv. of categories $\{\text{p-div}/O_C\} \longleftrightarrow \{(\Lambda, W) \mid \Lambda \text{ fin. free } \mathbb{Z}_p\text{-module}, W \subseteq \Lambda \otimes_{\mathbb{Z}_p} C \text{ subvector space}\}$

$$G \longmapsto (\Lambda = T(G),\ W = \mathrm{Lie}\, G \otimes_{O_C} C \overset{?}{\subseteq} \Lambda \otimes_{\mathbb{Z}_p} C)$$

## §1. Some preparations and notations

(1) Universal cover of a p-div gp. $R$ p-torsion ring, $G/R$ p-div gp.

define sheaf in $\mathbb{Q}_p$-vec. sp. on $\mathrm{Nilp}_R^{op}$: $R$-algebras on which $p$ is nilpotent by

$$\tilde{G} : S \in \mathrm{Nilp}_R^{op} \longmapsto \varprojlim_{p: G \to G} G(S).$$

note: $\tilde{G}$ depends only on isogeny class of $G$.

Let $S \twoheadrightarrow R$ be nilpotent thickening, then $\{\text{p-div}/S\}/\text{isog.} \xleftrightarrow{\text{bijection}} \{\text{p-div}/R\}/\text{isog.}$

consequence: $\tilde{G}$ lifts to $S$ and if $G_S$ is a p-div. gp/$S$ lifting $G$ then $\widetilde{G_S} = \tilde{G}$ and $\tilde{G}_S(S) = \tilde{G}(R)$. (use $\tilde{G}(R) = \mathrm{Hom}_R(\mathbb{Q}_p/\mathbb{Z}_p, G)[\frac{1}{p}]$).

remark: should picture $\tilde{G}$ as follows:

if $G$ connected, $\mathrm{Lie}\, G$ free of dim $d$ $\Rightarrow G = \mathrm{Spf}\, R[[T_1, \dots, T_d]]$

$\Rightarrow \tilde{G} = \mathrm{Spf}\, R[[T_1^{1/p^\infty}, \dots, T_d^{1/p^\infty}]]$

(2) The logarithm of p-div gps

Let $R$ be $\mathbb{Z}_p$-alg. with fin. gen. ideal of definition $I$ with $p \in I$, and $G/R$ a p-div. gp.

The generic fibre $G_\eta^{ad}$ of $G$ is the functor on the category affinoid $(R[\frac{1}{p}], R)$-algebras.

$$(S, S^+) \longmapsto \varinjlim_{S_0 \subseteq S^+} \varprojlim_n G(S_0/p^n)$$

(technical reasons behind it) — limit over all open and bounded subrings.

Prop. There exists a natural $\mathbb{Z}_p$-linear map of schemes:

$$\log_G : G_\eta^{ad} \longrightarrow \underbrace{\mathrm{Lie}\, G \otimes \mathbb{G}_a}_{(S, S^+) \longmapsto \mathrm{Lie}\, G \otimes_R S)^\#}$$

(+ some conditions on $\log_G$ that we omit)

Idea of proof: need to define $\log_G : G(R) \to \mathrm{Lie}\, G[\frac{1}{p}]$ (in a functorial way).

by Grothendieck-Messing theory there is $\log_G : \ker(G(R) \to G(R/p^2)) \xrightarrow{\simeq} p^2 \,\mathrm{Lie}\, G$.

it follows that $\forall_{x \in G(R)}\ \exists_{n \gg 0}$ s.t. $[p^n]_G(x) \in \ker(G(R) \to G(R/p^2))$

after dividing by $p^n$ it gives the desired map. □

Corollary: (i) $0 \to G_\eta^{ad}[p^\infty] \to G_\eta^{ad} \to \mathrm{Lie}\, G \otimes \mathbb{G}_a$

(ii) $G_\eta^{ad}$ is representable.

sketch: (i) follows from the construction.

(ii) $\mathrm{Lie}\, G \otimes \mathbb{G}_a$ is representable and $\log_G$ is an isom. in some neighb. of $0$. □

remark: should picture it as follows:

If $G$ is connected, $\mathrm{Lie}\, G$ free of dim $d$, $G = \mathrm{Spf}\, R[[T_1, \dots, T_d]]$

$$\rightsquigarrow G_\eta^{ad} \underset{\substack{\uparrow \\ \text{as a space.}}}{=} \mathring{\mathbb{B}}^d_{R[\frac{1}{p}]} \leftarrow \text{open unit ball of dim } d \text{ over } \mathrm{Spa}(R[\tfrac{1}{p}], R)$$

Passing to $\varprojlim$ yield $0 \to V(G)_\eta^{ad} \to \tilde{G}_\eta^{ad} \to \mathrm{Lie}\, G \otimes \mathbb{G}_a$ is exact seq. of sheaves on the cat. of affinoid algebras

where $T(G)$ is a Tate module (regarded as a functor), $V(G) = T(G)[\frac{1}{p}]$.

and $V(G)_\eta^{ad}$ is defined as $G_\eta^{ad}$ (functorially...)

## §2. From p-div gps to vector bundles

Given $C$, $\mathcal{O}_C$, $\mathcal{O}_C/p$, we can construct $A_{cris} = A_{cris}(\mathcal{O}_C/p)$, $B_{cris}^+ = A_{cris}[\frac{1}{p}]$.

Define $X = \mathrm{Proj} \bigoplus_{d \geq 0} (B^+_{cris})^{\varphi = p^d}$ the Fargues-Fontaine curve.

Thm. There exists a fully faithful embedding:

$$\{p\text{-div. gps}/(\mathcal{O}_C/p)\}_{/\text{up to isog.}} \xhookrightarrow{M(-)} \{\text{Dieudonné modules}/B^+_{cris}\} \longrightarrow \{\text{vector bundles on } X\}$$

$$G \longmapsto (M(G), \varphi)$$

$$(M, \varphi) \longmapsto \widetilde{\Big(\bigoplus_{d \geq 0} M^{\varphi = p^d}\Big)}$$

$$G \longmapsto \mathcal{E}(G).$$

Essential image consists of vec. bundles $\mathcal{E}$ on $X$ of slope $\lambda \in [0, 1]$.

**Remark**: This has the following application: $p$-div. gps over $O_C$ are <u>isotrivial</u> i.e.:

$\exists\, H/\overline{\mathbb{F}}_p$ $p$-div. gp and a quasi-isogeny $\rho: G\otimes O_C/p \longrightarrow H\otimes_{\overline{\mathbb{F}}_p} O_C/p$

We have a description of all v.b. on $X$: $\mathcal{E} = \bigoplus_{\lambda\in\mathbb{Q}} O_X(\lambda)^{d_\lambda}$ and one uses Dieudonné–Manin classification.

Let $G$ be a $p$-div. gp $/O_C$. $G_0 = G\otimes_{O_C} O_C/p$. We have:

$$\begin{array}{ccccccccc} 0 & \to & T[\frac{1}{p}] & \to & \tilde{G}(O_C) & \to & \mathrm{Lie}\,G\otimes C & \to & 0 \\ & & \| & & \|(*) & & \| & & \\ 0 & \to & T[\frac{1}{p}] & \to & M[\frac{1}{p}]^{\varphi=p} & \to & \mathrm{Lie}\,G\otimes C & \to & 0 \end{array}$$

exact as $C$ is alg. closed

$T$ = Tate module of $G(O_C)$.

where $M = M(G_0)$.

<u>ad $(*)$</u>: we have $\tilde{G}(O_C) = \tilde{G}(O_C/p) = \mathrm{Hom}_{O_C/p}(\mathbb{Q}_p/\mathbb{Z}_p, G)[\frac{1}{p}] = M[\frac{1}{p}]^{\varphi=p}$

<u>claim</u>: this exact sequence is identified with the global sections of a sequence:

$$0 \to T\otimes_{\mathbb{Z}_p} O_X \to \mathcal{E}(G_0) \to i_{\infty *}(\mathrm{Lie}\,G\otimes C) \to 0$$

of coherent sheaves on $X$. ($i_\infty: \{\infty\} \hookrightarrow X$ where $\infty \leftrightarrow C$)

We have an adjunction morphism: $\mathcal{E} \longrightarrow i_{\infty,*}\, i_\infty^*\, \mathcal{E}$

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & i_{\infty,*} i_\infty^* \mathcal{E} \\ \downarrow{\scriptstyle\text{global sections}} & & \downarrow \\ \tilde{G}(O_C) & \longrightarrow & M(G)\otimes C \\ \cup & & \cup \\ T\otimes C & \xrightarrow{\alpha_G} & (\mathrm{Lie}\,G^\vee)^*\otimes C \end{array}$$

Recall the G.-M. theory gives a submodule $(\mathrm{Lie}\,G^\vee)^*\otimes C \hookrightarrow M(G)$

<u>Claim</u>: this is a surjection.

There's a different description of $\alpha_G$. We have a map:

$\mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} T \longrightarrow G$ map of $p$-div. gps $/O_C$ $\overset{\text{dual}}{\Longrightarrow}$ $G^\vee \longrightarrow \mu_{p^\infty}\otimes_{\mathbb{Z}_p} T^\vee$ $\Rightarrow$

$\underset{\mathrm{Lie}(-)}{\Rightarrow}$ $\mathrm{Lie}\,G^\vee\otimes C \longrightarrow T^\vee\otimes C$ $\underset{\text{dual}}{\Rightarrow}$ $T\otimes C \longrightarrow (\mathrm{Lie}\,G^\vee)^*\otimes C$.

This is exactly $\alpha_G$.

**Thm.** $G \longmapsto \big( \underset{\overset{\shortparallel}{W}}{\mathrm{Lie}\, G \otimes C} \xhookrightarrow[\alpha^*_{G^\vee}]{} \underset{\overset{\shortparallel}{\Lambda \otimes C}}{T \otimes C} \big)$ induces the claimed equivalence of categories.

*proof*: (i) fully faithfulness. Let $G/\mathcal{O}_C$, we will show how to recover it from the data given above. Let $H = T(G) \otimes \mu_{p^\infty}$. We have a map $G \to H$ inducing $T(G) = T(H)$. Hence also $\mathrm{Lie}\, G \to \mathrm{Lie}\, H = T(G) \otimes \mathcal{O}_C$. This map is $\alpha^*_{G^\vee}$.

Claim:

this is cartesian!

$$\begin{tikzcd} 0 \arrow[r] & G_\eta^{ad}[p^\infty] \arrow[r] \arrow[d, equal] & G_\eta^{ad} \arrow[r] \arrow[d] & \mathrm{Lie}\, G \otimes \mathbb{G}_a = W \arrow[d, phantom, "\cap\shortparallel \alpha^*_{G^\vee}"] \\ 0 \arrow[r] & H_\eta^{ad}[p^\infty] \arrow[r] & H_\eta^{ad} \arrow[r, "\log_H"'] & \mathrm{Lie}\, H \otimes \mathbb{G}_a = \Lambda \otimes C \end{tikzcd}$$

sketch: let $x \in H_\eta^{ad}$ s.t. $\log_H(x) \in W \Rightarrow p^n x \in G_\eta^{ad}$ as $\log_G$ is an isom. in some neighb. of 0.

and
$$\begin{tikzcd} G_\eta^{ad} \arrow[r, "p^n"] \arrow[d] & G_\eta^{ad} \arrow[d] \\ H_\eta^{ad} \arrow[r] & H_\eta^{ad} \end{tikzcd}$$
is cartesian.

It follows that we can reconstruct $G$ from $(\Lambda, W)$: have constructed $G_\eta^{ad} \rightsquigarrow G = \coprod_{\substack{Y \in G_\eta^{ad} \\ \text{conn. comp.}}} \mathrm{Spf}\, H^0(Y, \mathcal{O}^+_{G_\eta^{ad}})$

This proves fully faithfulness by pulling back $\underset{\Lambda\otimes C \to \Lambda' \otimes C}{W \to W'}$ to $G_\eta^{ad}$-level.

(ii) left to show ess. surjectivity: similarly as above, we only need to construct $G_\eta^{ad}$.

Given $\Lambda, W \subseteq \Lambda \otimes_{\mathbb{Z}_p} C$, put $H = \mu_{p^\infty} \otimes \Lambda$. We have:

$$\begin{tikzcd} G_\eta^{ad} \arrow[r] \arrow[d] & W \otimes_C \mathbb{G}_a \arrow[d] \\ H_\eta^{ad} \arrow[r, "\log"] & \Lambda \otimes_{\mathbb{Z}_p} \mathbb{G}_a \end{tikzcd}$$

we define $G_\eta^{ad}$ like that.

**Thm.** Let $C$ be spherically complete and a valuation $C \xrightarrow{|\cdot|} \mathbb{R}_{\geq 0}$ is surjective. Then $(G_\eta^{ad})^\circ \cong \mathring{\mathbb{B}}^d$ open unit ball.

*Proof*: Step 1:

$$\begin{tikzcd} & G_{\eta, r}^{ad} \arrow[r, phantom, "\subseteq"] \arrow[d] & G_\eta^{ad} \arrow[d] \\ \text{closed ball of radius } r<1 \arrow[r] & \mathbb{B}_r^h \arrow[r, "\subseteq"] & \mathring{\mathbb{B}}^h = H_\eta^{ad} \end{tikzcd}$$

cartesian

Then $G_{\eta, r}^{ad} \cong \mathbb{B}^d$.

open unit ball

(in fact $\mathbb{B}_r^h \cong \mathbb{B}_{r'}^h$ for any $r, r'$.)

proof of step 1: firstly observe that $G^{ad}_{\eta,r} = \mathrm{Spa}(R, R^{\circ})$ affinoid and reduced $C$ alg. closed $\Rightarrow R^{\circ}$ top. of finite type. Let $\kappa$ = residue field of $O_C$. Then $R^{\circ} \otimes_{O_C} \kappa$ is reduced; $\mathrm{Spec}(R^{\circ} \otimes_{O_C} \kappa)$ is affine connected, reductive group scheme $\underset{\text{classification of such schemes}}{\Longrightarrow}$ it is an extension of a torus by unipotent group.

As mult. by $p$ is nilpotent $\Rightarrow$ torus part is trivial. Hence $\mathrm{Spec}(R^{\circ} \otimes_{O_C} \kappa)$ is a unipotent group hence isomorphic to $\mathrm{Spec}(\kappa[[T_1, \dots, T_d]]) \Rightarrow$

$\Rightarrow R^{\circ} = O_C \langle T_1, \dots, T_d \rangle$ so $G^{ad}_{\eta,r} \cong \mathbb{B}^d$.
(some work to be done here)

Lemma: $G^{ad}_{\eta} = \bigcup_{0<r<1} X_r$ (where $X_r = G^{ad}_{\eta,r} \cong \mathbb{B}^d$) s.t. $X_r \Subset X_{r+\varepsilon}$ and let $I_r \triangleleft A_r = H^0(X_r, O^+_{X_r}) = O_C \langle T_{i,r} \rangle$ ideal $(T_{1,r}, \dots, T_{d,r})$. and $M_r = I_r / I_r^2$ and $M = \bigcap M_r$. If $M \otimes C = M_r \otimes C$ $\forall r$ then $G^{ad}_{\eta} \cong \mathring{\mathbb{B}}^d$.

remark: assumptions of the lemma are fulfilled in our case are deduced from $H^{ad}_{\eta} = \mathring{\mathbb{B}}^h$.

sublemma: $V$ fin. dim. $C$-v.s. $N_1 \supseteq N_2 \supseteq \dots$ seq. of $O_C$-lattices s.t. $(\bigcap N_i) \otimes C = V$. then (i) $\bigcap N_i$ is a lattice
(ii) $R^1 \varprojlim N_i = 0$

proof of sublemma: Proceed by induction. Let $\dim V = 1$. Either $\bigcap N_i \cong \mathcal{O}$ or $\bigcap N_i \cong \mathfrak{m}$
if $\bigcap N_i \cong \mathfrak{m}$ then $\mathcal{O} \subseteq N_i$ $\forall i \Rightarrow \mathcal{O} \subseteq N_i$ contradiction.

we want $R^1 \varprojlim N_i = 0$, one can assume that $N_i = p^{-\varepsilon_i} \mathcal{O}$, $\bigcap N_i = \mathcal{O}$; $R^1 \varprojlim N_i = 0$ means that given $x_1, x_2, \dots \in \mathcal{O}$, $\exists y_1, y_2, \dots \in \mathcal{O}$ s.t. $x_i = y_i - p^{\varepsilon_i - \varepsilon_{i+1}} y_{i+1}$.

Using that $C$ spherically complete $\exists Y_1$ s.t. $Y_{i+1} = \frac{1}{p^{\varepsilon_1 - \varepsilon_{i+1}}}(Y_1 - X_1 - p^{\varepsilon_1 - \varepsilon_2} x_2 - \dots \dots p^{\varepsilon_1 - \varepsilon_i} x_i) \in \mathcal{O}$

this gives a sublemma for $\dim = 1$.

Induction step: use

$$\begin{array}{ccccccccc} 0 & \to & V' & \to & V & \to & V'' & \to & 0 \\ & & \cup & & \cup & & \cup & & \\ 0 & \to & N_i' & \to & N_i & \to & N_i'' & \to & 0 \end{array}$$

$\square$

<u>proof of lemma</u>: In particular, $M = \bigcap_{0<r<1} M_r = \bigcap_{0<r<1} I_r/I_r^2$ is free of dim $d$, $M \cong \mathcal{O}^d$.

and $R^1\varprojlim M_r = 0$. Hence, inductively, we have:

$$\varprojlim_r I_r^m/I_r^{m+1} = \mathrm{Sym}^m M \quad \text{and} \quad R^1\varprojlim_r I_r^m/I_r^{m+1} = 0.$$

Moreover $\varprojlim_r A_r/I_r = \mathrm{Sym}^m M / I_0^m$ with $R^1\varprojlim(-) = 0$.

Hence $\varprojlim_{r,m} A_r/I_r^m \cong (\mathrm{Sym}^\bullet M)^\wedge \cong \mathcal{O}_C[[T_1, \dots, T_d]]$

We are left to show that $H^0(G_\eta^{ad}, \mathcal{O}^+_{G_\eta^{ad}}) \cong \varprojlim_{r,m} A_r/I_r^m$.

$$\begin{array}{c} H^0(G_\eta^{ad}, \mathcal{O}^+_{G_\eta^{ad}}) \\ \parallel \\ \varprojlim_r H^0(X_r, \mathcal{O}^+_{X_r}) = \varprojlim_r \mathcal{O}_C\langle T_{1,r}, \dots, T_{d,r}\rangle. \end{array}$$

but $\varprojlim_{r,m} A_r/I_r^m = \varprojlim_r \mathcal{O}_C[[T_{1,r}, \dots, T_{d,r}]]$ and

$$\begin{tikzcd} \mathcal{O}_C\langle T_{1,r+\epsilon}, \dots, T_{d,r+\epsilon}\rangle \arrow[r] \arrow[d] & \mathcal{O}_C\langle T_{1,r}, \dots, T_{d,r}\rangle \\ \mathcal{O}_C[[T_{1,r+\epsilon}, \dots, T_{d,r+\epsilon}]] \arrow[ur, dotted] & \end{tikzcd}$$

← uniquely extends as $X_r \Subset X_{r+\epsilon}$.

and hence $\varprojlim_r \mathcal{O}_C\langle T_{1,r}, \dots, T_{d,r}\rangle = \varprojlim_r \mathcal{O}_C[[T_{1,r}, \dots, T_{d,r}]]$

$\square$