Adic Spaces

1 Affinoid rings

We fix k a non archimedean field, that is to say, k is a topological field, whose topology can be defined by a rank 1 valuation, and R will be a topological ring. In fact in the context of perfectoid spaces and rigid geometry, the topological ring R will always be a topological k-algebra, i.e. the topology of R will be compatible with the topology of k. Actually, most of the time, R will even be a normed k-algebra.
Recall that if R is a topological ring, we say that an ideal I defines the topology of R, if $\{I^n\}_{n\in\mathbb{N}}$ is a basis of neighbourhood of 0, and in that case R is called an adic ring.

Definition 1.1. Let R be a topological ring.

1. R is f-adic if there exists R_0 an open subring such that the topology of R_0 is defined by a finitely generated ideal I of R_0.

2. R is called a Tate ring if it is a f-adic ring and if there exists an invertible element which is topologically nilpotent.

Definition 1.2. A subset $M \subset R$ is said to be bounded if for all neighbourhood of 0, U, there exists V a neighbourhood of 0 such that $M.V \subseteq U$. An element $a \in R$ is said to be power-bounded if $\{a^n \mid n \in \mathbb{N}\}$ is bounded. We denote by R° the set of power-bounded elements.

Remark 1. If R is a normed k-algebra, one checks that $a \in R$ is power-bounded if and only if $\{\|a^n\| \mid n \in \mathbb{N}\}$ is bounded.

Definition 1.3.

1. A valuation on R is defined by a map $v : R \to \Gamma \cup \{0\}$ where Γ is a totally ordered commutative abelian group (noted multiplicatively), such that $v(ab) = v(a)v(b)$, $v(a+b) \leq \max(v(a), v(b))$, $v(0) = 0$ and $v(1) = 1$. The order on $\Gamma \cup \{0\}$ is defined by the order of Γ and the fact that 0 is a minimum element, moreover, we set $0 \cdot \gamma = 0$ for all $\gamma \in \Gamma$. To simplify notations, we'll often denote valuations by $|\cdot|$.

2. $|\cdot| : R \to \Gamma \cup \{0\}$ is said to be a continuous valuation if for all $\gamma \in \Gamma$, there exists U a neighbourhood of 0 such that $|U| \subseteq [0, \gamma[= \{\alpha \in \Gamma \cup \{0\} \mid \alpha < \gamma\}$.

3. The subgroup of Γ generated by $|R| \setminus \{0\}$ is called the value group of $|\cdot|$ and is denoted by $\Gamma_{|\cdot|}$.

4. Two valuations v and w on R are called equivalent if there exists an isomorphism of totally ordered groups : $\alpha : \Gamma_v \simeq \Gamma_w$ such that for all $a \in R$, $w(a) = \alpha(v(a))$.

If $|\cdot|$ is a valuation, $\mathrm{supp}(|\cdot|) = \{a \in R \mid |a| = 0\}$ is a prime ideal of R and $|\cdot|$ induces a valuation on the fraction field K of $R/\,\mathrm{supp}(|\cdot|)$. One can check that two valuations v and w are equivalent if and only if $\mathrm{supp}(v) = \mathrm{supp}(w)$ and the valuation rings they define on K are the same. This is also equivalent to say that for all $a, b \in R$, $v(a) \leq v(b)$ if and only if $w(a) \leq w(b)$.

Definition 1.4. An affinoid ring is given by a pair (R, R^+) where R is a f-adic ring, and $R^+ \subseteq R^\circ$ is an open and integrally closed subring of R.[1]
A morphism f of affinoid rings between (R, R^+) and (S, S^+) is a continuous morphism $f : R \to S$ such that $f(R^+) \subseteq S^+$.

Remark 2. According to the generality of the presentations, the definitions of [5] ,[6],[2] might seem different but actually agree. Indeed let R be a topological k-algebra, the following propositions are equivalent :

1. R is f-adic
2. R is a Tate ring
3. There exists a subring R_0 such that aR_0, $a \in k^\times$ forms a basis of open neighbourhoods of 0.[2]

If R is in fact a normed k-algebra, R is automatically a Tate k-algebra.

Definition 1.5. Let (R, R^+) be an affinoid ring. One defines

$$X = \mathrm{Spa}(R, R^+) = \{|\cdot| : R \to \Gamma \cup \{0\} \text{ continuous valuations such that } |R^+| \leq 1\} / \simeq$$

where $\simeq$ denotes the equivalence relation on valuations defined above.
We equip X with the topology generated by the open subsets $\{|\cdot| \in X \mid |a| \leq |b| \neq 0\}$ where $a, b \in R$.

If $x \in \mathrm{Spa}(R, R^+)$, then x is a valuation $x : R \to \Gamma \cup \{0\}$, and if $f \in R$, we'll denote $x(f)$ by $|f(x)|$.[3]

Definition 1.6. A topological space X is said to be spectral if X is quasi-compact, has a basis of the topology made of quasi-compact open which is stable under finite intersection, and such that every irreducible closed subset has a unique generic point.

[1] Huber usually denotes an affinoid ring by $(A^\triangleright, A^+)$ and calls the subrings of $A^\triangleright$ having the property of A^+ rings of integral elements.
[2] This corresponds to the definition of a Tate k-algebra in [6, Def 2.5].
[3] With these notations, x is the point of a space, and f a function on this space.

In [3] it is proved that a topological space X is spectral if and only if it is isomorphic to $\mathrm{Spec}(B)$ for some ring.

Proposition 1.7. *Let (R, R^+) be an affinoid ring, and $X = Spa(R, R^+)$. X is a spectral space. Moreover define a rational subset of X as $U(\frac{f_1,\ldots,f_n}{g}) = \{x \in X \mid |f_i(x)| \leq |g(x)| \neq 0 \; i = 1 \ldots n\}$ where the f_i's define an open ideal of R. Then the rational subsets form a basis of neighbourhood of X, stable under finite intersection.*

Spa then defines a functor from the category of affinoid ring to the category of topological spaces.
When R is a Tate ring, R is the only open ideal of R, so in that case, the condition *the f_i's generate an open ideal* is equivalent to *the f_i's generate* R, and one checks that in that case $\{x \in X \mid |f_i(x)| \leq |g(x)| \neq 0 \; i = 1 \ldots n\} = \{x \in X \mid |f_i(x)| \leq |g(x)| \; i = 1 \ldots n\}$.
We list some examples:

1. If $\mathcal{A}$ is an affinoid k-algebra (in the sense of rigid geometry [1]) i.e. a quotient of the Tate algebra $k\langle T_1, \ldots, T_n\rangle$, then $(\mathcal{A}, \mathcal{A}^\circ)$ is an affinoid ring[4] in the above sense and we have the following result:

   Theorem 1.8 ([4]). *Let $\mathcal{A}$ be an affinoid k-algebra. Then the topos associated to X =$Spa(\mathcal{A}, \mathcal{A}^\circ)$ is equivalent to the topos of the rigid space $Max(\mathcal{A})$.*

2. If R is ring, I is an ideal of R such that R is complete for the topology defined by I, then (R, R) is an affinoid ring.

3. Let $R = k\langle T^{1/p^\infty}\rangle$ and $R^+ = k^\circ\langle T^{1/p^\infty}\rangle$. Then (R, R^+) is an affinoid ring. If $x = (x_i)_{i\geq 1}$ is a sequence of points of $\overline{k}$ such that $x_{i+1}^p = x_i$ for all $i \geq 1$, and $x_1 \in \overline{k}^\circ$, then we can define a morphism of k-algebra $x : k\langle T^{1/p^\infty}\rangle \to \overline{k}$ defined by $T^{1/p^n} \mapsto x_n$, and

$$\begin{array}{rccc} |\cdot|_x : & k\langle T^{1/p^\infty}\rangle & \to & \mathbb{R} \\ & f & \mapsto & |f(x)| \end{array}$$

   is a point of $\mathrm{Spa}(k\langle T^{1/p^\infty}\rangle, k^\circ\langle T^{1/p^\infty}\rangle)$.

2 The Example of the closed unit disc

If $\mathcal{A}$ is an affinoid k-algebra, we can then consider $\mathrm{Max}(\mathcal{A})$, $\mathcal{M}(\mathcal{A})$ the Berkovich space associated to $\mathcal{A}$, and X= $\mathrm{Spa}(\mathcal{A}, \mathcal{A}^\circ)$. Then, as a set, $\mathcal{M}(\mathcal{A})$ corresponds to the maximal points of $\mathrm{Spa}(\mathcal{A}, \mathcal{A}^\circ)$ [5]. We now explain in details the topological space corresponding to the closed unit disc $\mathbb{B}$ =$\mathrm{Spa}(k\langle T\rangle, k^\circ\langle T\rangle)$ when $k = \overline{k}$. First we must find the points of the corresponding Berkovich space, namely:

[4]One must check that $\mathcal{A}^\circ$ is integrally closed in $\mathcal{A}$.
[5]A maximal point is a point that has no proper generalization.

Points of type 1, that correspond to maximal ideals of $k\langle T\rangle$. They correspond to k° according to the Nullstellensatz.
Points of type 2, corresponding to closed disc of rational radius. Precisely, if $r \in |k|$, and $x \in k^\circ$, then we can associate to it a point $\eta_{x,r} \in \mathbb{B}$ such that for $f \in k\langle T\rangle$, and $f = \sum_n a_n(T-x)^n$,

$$|f(\eta_{x,r})| = \max_n |a_n| r^n = \max_{z\in B} |f(z)| = |f|_B$$

where we set $B = \{z \in k^\circ \mid |z-x| \leq r\}$. If x and x' define the same ball of radius r, i.e. $|x-x'| \leq r$, then the two associated points will coincide.
Points of type 3 are defined similarly as points of type 2, except that $r \notin |k|$.
Points of type 4 correspond to decreasing sequences $\{B_i\}$ of closed ball of k° with empty intersection, and then $\eta \in X$ is defined by $|f(\eta)| = \inf_i |f|_{B_i}$.
These four types of points correspond to rank 1 valuations, and were already in the Berkovich unit disc.
In $\mathbb{B}$ there are new points, say type 5 points, which are rank 2 valuations: define $\Gamma = \mathbb{R}_{>0} \times \mathbb{Z}$ with the lexicographic order, that we note multiplicatively, so that we see Γ as $\mathbb{R}_{>0} \times \gamma^{\mathbb{Z}}$, with $\gamma < 1$. So for this order of Γ, $\gamma < 1$, but for all $r \in \mathbb{R}_{>0}$, if $r < 1$ then $r < \gamma$. Now, if $x \in k^\circ$ and $r \leq 1$ we define a valuation:

$$\begin{array}{cccc} \eta_{x,r^-}: & k\langle T\rangle & \to & \Gamma \\ & \sum_n a_n(T-x)^n & \mapsto & \max |a_n| r^n \gamma^n \end{array}$$

One can check that if $r \notin |k|$ this valuation is equivalent to the point of type 3 associated to x and r. Otherwise, this defines a new valuation. Somehow, this is the valuation corresponding to the open ball centered in x of radius r. If $|x-x'| < r$, then $\eta_{x,r^-} = \eta_{x',r^-}$.
Actually, if we define $\Gamma' = \mathbb{R}_{>0} \times \gamma^{\mathbb{Z}}$ with the lexicographic order such that $\gamma > 1$ we define

$$\begin{array}{cccc} \eta_{x,r^+}: & k\langle T\rangle & \to & \Gamma \\ & \sum_n a_n(T-x)^n & \mapsto & \max |a_n| r^n \gamma^n \end{array}$$

This defines a point of $\mathbb{B}$ if $r < 1$. If $r = 1$, $\eta_{x,1^+}$ is well continuous but doesn't satisfy $\eta_{x,1^+}(k^\circ\langle T\rangle) \leq 1$.

Remark 3. If one defines $C = \{\sum_n a_n T^n \mid |a_0| \leq 1 \text{ and } |a_i| < 1,\ i > 0\}$, then $(k\langle T\rangle, C)$ is an affinoid ring, and $\mathrm{Spa}(k\langle T\rangle, C) = \mathbb{B} \cup \{\eta_{0,1^+}\}$.

Finally we make the following remark: if $x \in k^\circ$ and $r < 1$ with $r \in |k|$, then for all y such that $|y-x| \leq r$, $\eta_{y,r^-} \in \overline{\{\eta_{x,r}\}}$, and $\eta_{x,r^+} \in \overline{\{\eta_{x,r}\}}$ and this describes all the points of $\overline{\{\eta_{x,r}\}}$. If $r = 1$ then for all $y \in k^\circ$, $\eta_{y,1^-} \in \overline{\{\eta_{x,1}\}}$. Hence for $r < 1$ $\overline{\{\eta_{x,r}\}}$ is isomorphic to $\mathbb{P}^1_\kappa$ and $\overline{\{\ \eta_{x,1}\}}$ is isomorphic to $\mathbb{A}^1_\kappa$ where $\kappa = \tilde{k}$ is the residue field of k.

3 (Pre)-Sheaves

Let $X = \mathrm{Spa}(R, R^+)$ and $U = U(\frac{f_1,\ldots,f_n}{g})$ be a rational subset. We consider B the integral closure of $R^+[\frac{f_1}{g},\ldots,\frac{f_n}{g}]$ in $R[\frac{f_1}{g},\ldots,\frac{f_n}{g}]$. Then, $(R[\frac{f_1}{g},\ldots,\frac{f_n}{g}], B)$

is an affinoid ring. We then take its completion $R\langle\frac{f_1}{g},\ldots,\frac{f_n}{g}\rangle,\hat{B})$.[6] By functoriality, one can define a morphism

$$\psi : \mathrm{Spa}(R\langle\frac{f_1}{g},\ldots,\frac{f_n}{g}\rangle,\hat{B}) \to \mathrm{Spa}(R,R^+)$$

It fulfils the following universal property : for all complete affinoid ring (S,S^+) and $\varphi : (R,R^+) \to (S,S^+)$ a morphism of affinoid ring such that Im $(\mathrm{Spa}(\varphi)) \subseteq U$ where $\mathrm{Spa}(\varphi) : \mathrm{Spa}(S,S^+) \to \mathrm{Spa}(R,R^+)$, then φ factorizes uniquely through $(R\langle\frac{f_1}{g},\ldots,\frac{f_n}{g}\rangle,\hat{B})$.
From this it follows that $R\langle\frac{f_1}{g},\ldots,\frac{f_n}{g}\rangle$ and $\hat{B}$ depend only on U. We then set $\mathcal{O}_X(U) = R\langle\frac{f_1}{g},\ldots,\frac{f_n}{g}\rangle$ and $\mathcal{O}_X^+(U) = \hat{B}$. In this way, one checks that $\mathcal{O}_X$ and $\mathcal{O}_X^+$ are presheaves on the rational subsets of X. Now, if $W \subseteq X$ is an open subset, we set

$$\mathcal{O}_X(W) = \varprojlim_{U\subseteq W} \mathcal{O}_X(U)$$

where the limit is taken over the rational subsets $U \subseteq W$, and likewise for $\mathcal{O}_X^+$. They are presheaves of complete topological rings.
One checks that for all $x \in X$, and $x \in U$ a rational subset, $x : R \to \Gamma$ can be extended to $x : \mathcal{O}_X(U) \to \Gamma$. So x can be extended to $\mathcal{O}_{X,x}$. $\mathcal{O}_{X,x}$ is a local ring with maximal ideal $M_x = \{f \in \mathcal{O}_{X,x} \mid |f(x)| = 0\}$. We set $k(x) = \mathcal{O}_{X,x}/M_x$. So $k(x)$ is naturally equipped with a valuation: $f \to |f(x)|$ and we set $k^+(x)$ its valuation ring. Huber proves that for an open subset U:

$$\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) \mid |f(x)| \leq 1 \ \forall x \in U\}$$

$\mathcal{O}_X$ is not in general a sheaf. The previous remark however shows that if it was a sheaf, then $\mathcal{O}_X^+$ would also be a sheaf. However

Definition 3.1. A topological ring is strongly noetherian if for all $n \in \mathbb{N}$, $R\langle T_1,\ldots,T_n\rangle$ is noetherian [7].

Proposition 3.2. *If R is strongly noetherian, and $X = Spa(R,R^+)$, then $\mathcal{O}_X$ is a sheaf.*

One then defines (V) as the category of locally ringed spaces $(X,\mathcal{O}_X)$, such that the sheaf $\mathcal{O}_X$ is a sheaf of topological rings, and such that for all $x \in X$, there is given an equivalence class of valuation v_x of the stalk $\mathcal{O}_{X,x}$. The morphisms in (V) must be compatible with all these data.

Definition 3.3. Let (R,R^+) be an affinoid ring and $X = \mathrm{Spa}(R,R^+)$. If $\mathcal{O}_X$ is a sheaf, we'll say it is an affinoid adic space (seen as an object of (V)). An adic space is an object of (V) which is locally an affinoid adic space.

Adic spaces fulfil most of the expected properties, such as

[6] One has to convince himself that the completion of arbitrary topological rings exists, and that the completion $(\hat{R},\hat{R^+})$ of an affinoid ring (R,R^+) is still an affinoid ring.

[7] I don't know an example of a noetherian ring R which is not strongly noetherian.

Proposition 3.4. *If X and $Spa(R, R^+)$ are adic spaces, then*

$$Hom_{adic\ sp.}(Y, Spa(R, R^+)) \simeq Hom_{aff.\ ring}((\hat{R}, R^+), (\mathcal{O}_X(X), \mathcal{O}_X^+(X)))$$

Theorem 3.5. *There is a functor*

$$r_k : \{ \text{ rigid spaces over } k\} \to \{\text{adic spaces over } Spa(k, k^\circ)\}$$

obtained by gluing the functor $Max(\mathcal{A}) \mapsto Spa(\mathcal{A}, \mathcal{A}^\circ)$. It is fully faithful and induces an equivalence of category :

$$\{\text{rigid spaces over } k \text{ quasi-separated}\} \simeq \{\text{quasi-separated adic spaces loc. of finite type over } Spa(k, k^\circ)\}$$

where finite type will be defined in the next section.

4 Finite type, proper, étale ... morphisms

For simplicity we assume that R is a Tate ring.

Definition 4.1. A morphism $f : (R, R^+) \to (S, S^+)$ between affinoid rings is a quotient map if f is surjective continuous and open, and S^+ is the integral closure of $f(R^+)$ in S.
If (R, R^+) is an affinoid ring with R Tate, we set

$$(R, R^+)\langle T_1 \dots T_n\rangle = (R\langle T_1 \dots T_n\rangle, R^+\langle T_1 \dots T_n\rangle)$$

which is an affinoid ring[8].

Definition 4.2. A morphism of affinoid rings $(R, R^+) \to (S, S^+)$ is of topologically finite type if it factorises as $(R, R^+) \to (R, R^+)\langle T_1 \dots T_n\rangle \xrightarrow{\pi} (S, S^+)$ where π is a quotient map.

Definition 4.3. Let $f : X \to Y$ be a morphism of adic spaces. It is

1. locally of weakly finite type if for all $x \in X$ there exists U, V, open affinoid subspaces of X, Y, such that $x \in U$, $f(U) \subseteq V$ and such that the morphism of f-adic ring $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is of topologically finite type.
2. locally of finite type if for all $x \in X$ there exists U, V, open affinoid subspaces of X, Y, such that $x \in U$ and $f(U) \subseteq V$ and such that the affinoid ring morphism $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is of topologically finite type.
3. f is of finite type if it is quasi-compact and locally of finite type.

Proposition 4.4. *If $f : X \to Z$ and $g : Y \to Z$ are morphisms of adic spaces with f locally of finite type, one can define the fiber product of f and g.*

[8]One has to check that $R^+\langle T_1 \dots T_n\rangle$ is integrally closed in $R\langle T_1 \dots T_n\rangle$.

Definition 4.5. If $f : X \to Y$ is a morphism of adic spaces locally of finite type [9], then

1. f is separated if $\Delta(X)$ is closed in $X \times_Y X$ where Δ is the diagonal morphism.
2. f is universally closed if it is locally of weakly finite type, and for all adic morphism [10] $Y' \to Y$, $X \times_Y Y' \to Y'$ is closed.
3. f is proper if it is of finite type, separated and universally closed.

In [5, 1.3] a kind of valuative criterion for properness is proved.
$\mathbb{B}$ =Spa$(k\langle T\rangle, k^\circ\langle T\rangle)$ is not proper, however, if $C = \{\sum_n a_n T^n \mid |a_0| \leq 1 \text{ and } |a_i| < 1,\ i > 0\}$, then Spa$(k\langle T\rangle, C)$ is proper.
This notion of properness is related to the notion of properness of rigid k-spaces, see [5, Rem 1.3.19].

Definition 4.6.

1. A morphism $f : (R, R^+) \to (S, S^+)$ of affinoid rings is finite if it is of topologically finite type, $R \to S$ is finite, and S^+ is the integral closure of $f(R^+)$.
2. A morphism of adic spaces $f : X \to Y$ is finite if for all $y \in Y$, there exists an affinoid neighbourhood U such that $f^{-1}(U) = V$ is affinoid and $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is finite.
3. f is locally quasi-finite if $\forall y \in Y$, $f^{-1}(y)$ is discrete.
4. f is quasi-finite if f is quasi-compact and locally quasi-finite.

Definition 4.7. If (R, R^+) is an affinoid ring, and I an ideal of R, we denote by $(R, R^+)/I$ the affinoid ring $(R/I, (R^+/(I \cap R^+))^c)$ where $(R^+/(I \cap R^+))^c$ is the integral closure of $R^+/(I \cap R^+)$ in R/I.[11]

Definition 4.8.

1. A morphism $f : X \to Y$ which is locally of finite type is called unramified (resp. smooth, resp. étale) if for all affinoid ring (R, R^+), and all ideal I of R such that $I^2 = \{0\}$, and $g : \text{Spa}(R, R^+) \to Y$, $\text{Hom}_Y(\text{Spa}(R, R^+), X) \to \text{Hom}_Y(\text{Spa}((R, R^+)/I), X)$ is injective (resp. surjective, resp. bijective).
2. A morphism $f : X \to Y$ is said to be unramified (resp. smooth, resp. étale) at $x \in X$ if there exist U, V open subsets of X, Y such that $x \in U$ and $f(U) \subseteq V$ and $f_{|U} : U \to V$ is unramified (resp. smooth, resp. étale).

[9] Huber needs only f to be locally of $^+$weakly finite type which is more general than locally of finite type.
[10] see p.46 of [5] for the definition of an adic morphism
[11] that we equip with the quotient topology.

One can define sheaf of differentials $\Omega_{X|Y}$ (which is an $\mathcal{O}_X$-module), when $f : X \to Y$ is locally of finite type, such that

Proposition 4.9.

1. *f is unramified if and only if $\Omega_{X|Y} = 0$.*

2. *If f is smooth, $\Omega_{X|Y}$ is a locally free $\mathcal{O}_X$-module.*

3. *If (R, R^+) is an affinoid ring with R Tate, $Y = Spa(R, R^+)$, then $f : X \to Y$ is smooth if and only if for all $x \in X$ there exists a commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{g} & Spa((R, R^+)\langle T_1, \dots, T_n\rangle) \\ \downarrow f & \swarrow h & \\ Spa(R, R^+) & & \end{array}$$

   where U is an open neighbourhood of x, h is the natural morphism and g is étale.

Proposition 4.10. *Let $f : X \to Y = Spa(R, R^+)$ be a morphism of affinoid adic spaces, with R Tate. The following are equivalent:*

1. *f is étale.*

2. *There exists $n \in \mathbb{N}$, $f_1 \dots f_n \in R\langle T_1 \dots T_n\rangle$ such that if I is the ideal $(f_1, \dots, f_n)$ then $det(\frac{\partial f_i}{\partial T_j})_{1 \leq i,j \leq n}$ is invertible in $R\langle T_1 \dots T_n\rangle / I$, and X is Y-isomorphic to $Spa((R, R^+)\langle T_1 \dots T_n\rangle / I)$.*

3. *There exists $n \in \mathbb{N}$, $f_1 \dots f_n \in R[T_1 \dots T_n]$ such that if I is the ideal $(f_1, \dots, f_n)$ then $det(\frac{\partial f_i}{\partial T_j})_{1 \leq i,j \leq n}$ is invertible in $R\langle T_1 \dots T_n\rangle / I$, and X is Y-isomorphic to $Spa((R, R^+)\langle T_1 \dots T_n\rangle / I)$.*

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