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(I) p-divisible groups

Def. S a scheme, $h \in \mathbb{N}$, a p-div. group of height h is $G_1 \to G_2 \to \dots \to G_n \to \dots$ where G_n is comm. group scheme $/S$ loc. free of rank $\mathrm{rk}(G_n) = p^{hn}$ such that $0 \to G_n \to G_{n+1} \xrightarrow{[p^n]} G_{n+1} \to 0$ is exact. $(\Rightarrow \forall_{n,m}\ 0 \to G_n \to G_{n+m} \xrightarrow{[p^n]} G_m \to 0)$

Example: * $G_n = \frac{\mathbb{Z}\frac{1}{p^n}}{\mathbb{Z}} = \frac{\mathbb{Z}_p\frac{1}{p^n}}{\mathbb{Z}_p} = \mathbb{Q}_p/\mathbb{Z}_p[p^n]$ $\rightsquigarrow$ constant p-div. group. (over any S)

* $\mu_{p^n} = \ker(\mathbb{G}_m \xrightarrow{[p^n]} \mathbb{G}_m)$

* abelian scheme $/S$, $A[p^n] = \ker(A \xrightarrow{[p^n]} A)$ $\quad (h = 2g,\ g = \dim_S A)$

* Cartier duality G is finite loc. free (of rk r) comm. gp scheme $\rightsquigarrow G' = \mathbb{D}(G)$ (rank r)
$\forall$ S-scheme T $\quad G'(T) = \mathrm{Hom}_{T\text{-gp sch}}(G_T, \mathbb{G}_{m,T})$. We have $G'' \cong G$.

If $G_n \to G_{n+1} \dots$ is p-div. of height h, $\rightsquigarrow G_n' \leftarrow G_{n+1}'$ and it is not clear that this gives a p-div gp.
but $G_{n+1} \xrightarrow{[p]} G_n \rightsquigarrow G_n' \to G_{n+1}'$ and this defines the dual p-div group. (of height h) too

Prop.: If (G_n) is a p-div gp, $\exists$ an exact sequence
$$0 \to G^0 \to G \to G^{\acute{e}t} \to 0 \quad \text{where } G^0 \text{ is connected and } G^{\acute{e}t} \text{ étale.}$$

Prop. If G is a p-div gp, then the following are equivalent:
(1) G is connected
(2) The augmentation ideal $I = \ker(\mathcal{O}_G \to \varepsilon_* \mathcal{O}_S)$ is locally nilpotent
(3) $\forall_{s\in S}$ G_s is connected
(4) $\forall_{s\in S}$ G_s has a single point (5) $\forall_{\bar{s}}$ $G_{\bar{s}}$ has a single point.

Remark: If p is invertible in $\mathcal{O}_S$, then p-div. groups are étale

Prop. S connected, ξ geom point of S, then there exists an equiv. of categories:
$$\{\text{étale p-div groups}\} \longleftrightarrow \pi_1(S,\xi)\text{-}, \mathbb{Z}_p\text{-module finite free.}$$
$$(G_n) \longmapsto \varprojlim_n G_n(\xi)$$

example: p invertible, $(\mu_{p^n})_n \longleftrightarrow \mathbb{Z}_p(1)$.

Prop.: if S is a $\mathbb{F}_p$-scheme and G is an étale p-div. group. Then G' is connected.

Def. G is bi-infinitesimal if G and G' are connected.

Cartier d.: étale $\xrightarrow{\mathbb{D}}$ conn. (multiplicative type)
bi-inf. is stabilized by duality.

$\mu_{p^n} \longleftrightarrow \mathbb{Z}/p^n\mathbb{Z}$

$\alpha_{p^n} = \ker(\mathbb{G}_{a,k} \xrightarrow{F^n} \mathbb{G}_{a,k})$ bi-inf

$\Rightarrow$ $0 \to (G')^0 \to G' \to (G')^{ét} \to 0$ hence $0 \to ((G')^{ét})' \to G \to (G')^0$

$G^{mult} \subset G^0 \subset G$. $\qquad$ $\overset{\shortparallel}{G^{mult}}$ = the largest subgp of mult. type

Prop.: If $S = \mathrm{Spec}(k)$, k perf then $G = G^{mult} \times \underset{\substack{\shortparallel \\ G^0/G^{mult}}}{G^{bi\text{-}inf}} \times G^{ét}$.

② connected p-divisible groups

Def. A n-dim, a formal Lie group is a formal group law of dim n:

$F: A[[X_1, \dots, X_n]] \to A[[Y_1, \dots, Y_n, Z_1, \dots, Z_n]]$ s.t. $\quad (X = (X_1, \dots, X_n))$

- (1) $F(X,0) = X = F(0,X)$ $\quad$ (2) $F(X, F(Y,Z)) = F(F(X,Y), Z)$
- (3) $F(X,Y) = F(Y,X)$.

exemple: $n=1$, $\mathbb{G}_a$: $Y+Z$; $\mathbb{G}_m$: $(1+Y)(1+Z)$

if A is $\mathbb{Q}$-alg, all 1-dim formal group are isom. to $\mathbb{G}_a$.

Def. If $\mathcal{G} = (A[[X]], F)$, we can define for $n \in \mathbb{N}$ $\; n: A[[X]] \to A[[X]]$ an endom. of $\mathcal{G}$ by: $[1] = \mathrm{Id}$, $[n+1](X) = F([n]_1 X, X)$.

$\mathcal{G}$ is p-divisible if $[p]: A[[X]] \to A[[X]]$ is finite.

ex $(\mathbb{G}_a)_{\mathbb{Z}_p}$ is not p-div.

Thm. R a noeth. loc. complete ring, with res. char. p:

$$\left\{\begin{matrix}\text{formal group laws} \\ \text{on } R \text{ which are } p\text{-div}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{connected} \\ p\text{-div. groups}_R\end{matrix}\right\}$$

$$\mathcal{G}_1 = (R[[X]], F) \longmapsto G_n = \mathrm{Spec}(R[[X]]/[p^n])$$

is an equiv. of categories

Cor. If G is p-div./R $\leadsto$ $\dim G = n$. If G is of height h $\dim G + \dim G' = h$.

Lubin-Tate theory

Thm. K local field, π a uniformiser of $\mathcal{O}_K$, $f(X) = \pi X + \dots \in \mathcal{O}_K[[X]]$ s.t. $f(X) = X^q \bmod \pi$ where $q = \#(\mathcal{O}_K/\pi)$. Then there exists a unique formal gp law $\mathcal{G}_\pi$ on $\mathcal{O}_K$ s.t. $f \in \mathrm{End}(\mathcal{G}_\pi)$ and $\exists\, \mathcal{O}_K \to \mathrm{End}(\mathcal{G}_\pi)$ s.t. $a \longmapsto [a]_{\mathcal{G}_\pi} = aX + \dots$

If $K/\mathbb{Q}_p$ finite, π unif, $p = \pi^e \cdot u$, $u \in \mathcal{O}_K$ then

$$[p]_{\mathcal{G}_\pi} \equiv u X^{q^e} \bmod \pi \equiv u X^{p^{ne}}$$

$\leadsto$ $\mathcal{G}_\pi$ correspond to a p-divisible group whose height is $ne = [K:\mathbb{Q}_p]$

(3.) Dieudonné module

k a perf. field of char p. $\rightsquigarrow$ $W(k) \circlearrowleft \sigma$ Frobenius

<u>Def</u>. A <u>Dieudonné module</u> M is a finitely gen. $W(k)$-module and $F : M \to M$ σ-lineare endomorphism. s.t. $F(\lambda m) = \sigma(\lambda) F(m)$, $\lambda \in W(k)$, $m \in M$
such that $M \supset F(M) \supset pM$.

<u>Theorem</u>: There is an antiequiv. of categories:
$$\{p\text{-div. groups}/k\} \longleftrightarrow \{\text{finite free Dieudonné modules}\}$$
$$G \longmapsto M(G)$$

* G is connected $\iff$ F is nilpotent (on $M(G)$)
* height $G = \dim_{W(k)} M(G)$.

<u>Cor.</u> $k = \overline{\mathbb{F}}_p$, $\forall_{h \geq 1}$ there exists a unique p-div. group of height h.

<u>Def</u>. $G \xrightarrow{f} H$ morph. of p-div. gp is an isogeny if $\ker f$ is finite and f is surjective.

$k/\mathbb{F}_p$ perfect field, $K = W(k)[\frac{1}{p}]$. An <u>isocrystal</u> is (M, F) where M is a finite dim. K-vec. space and $F : M \to M$ is σ-linear automorphism.

<u>Thm.</u> There is ~~an equivalence of categories~~ fully faithful functor.
$$\left\{\begin{matrix}p\text{-div. groups}\\ \text{up to isogeny}\end{matrix}\right\} \longrightarrow \{K\text{-isocrystals}\}$$

If $k = \bar{k}$ then the cat. of K-isocrystals is semi-simple with simple objects indexed by $\lambda = \frac{r}{s} \in \mathbb{Q}$, $s \in \mathbb{N}^*$, $(r,s) = 1$. ; <u>explicitly</u>: $K[F]/(F^s - p^r) = M_\lambda$ is simple.

(II) <u>Formal K-vector spaces</u>

(1.) <u>remarks on additive group</u>

k perf. field $\supset \mathbb{F}_q$, $K \cong \mathbb{F}_q((\pi))$, $\mathbb{G}_{a,k}$ can be seen as a functor: $k\text{-Alg} \longrightarrow \text{AbGp}$, $R \longmapsto (R, +)$, $k \longrightarrow \operatorname{End}(\mathbb{G}_{a,k})$.
$\lambda \in k$, $X \longmapsto \lambda X$.

in our case: $\tau_p : \mathbb{G}_{a,k} \longrightarrow \mathbb{G}_{a,k}$, $X \longmapsto X^p$

$\operatorname{End}(\mathbb{G}_{a,k}) = k\{\tau_p\} = \left\{\sum_{i=0}^{d} a_i \tau_p^i\right\}$
non-comm. if $a \in k$ then $\tau_p a = a^p \tau_p$

define $\operatorname{End}_{\mathbb{F}_q}(\mathbb{G}_{a,k}) = \left\{\begin{matrix}\varphi \in \operatorname{End}(\mathbb{G}_{a,k}) \mid \\ \varphi \circ \lambda = \lambda \circ \varphi, \lambda \in \mathbb{F}_q\end{matrix}\right\}$

if $q = p^n$, $\operatorname{End}_{\mathbb{F}_q}(\mathbb{G}_{a,k}) = k\{\tau\}$, $\tau = \tau_p^n$
$X \longmapsto X^q$.

Define: $\mathrm{Nil} : k\text{-Alg} \longrightarrow \mathrm{AbGp}$ ← nilpotent elts. Nil is <u>pro-representable</u>:

$R \longmapsto (\mathrm{Nil}(R), +)$

$\mathrm{Nil}(R) = \varinjlim_{n\geq 0} \mathrm{Hom}_k(k[X]/X^n, R)$; Nil is represented by the formal add. gp $\underbrace{\ }_{k[[X]]}$ Spf

Def.: $\mathrm{Nil}^b = \mathbb{G}^b_{a,k} : k\text{-Alg} \longrightarrow \mathrm{TopVect}_{\mathbb{F}_q}$ via $R \longmapsto \varprojlim_{\tau} \mathrm{Nil}(R) = \{(x_0, x_1, \dots)$ s.t. $x_0 \in \mathrm{Nil}(R)$ and $x_{i+1}^q = x_i\}$ with top. induced by prod top.

$\mathbb{G}^b_{a,k}(R) = \varinjlim_{n\geq 0} \mathrm{Hom}(k[X^{1/q^\infty}]/X^n, R)$ ["tilt"]

	$R \to ?$	ring	$\mathbb{F}_q$-endo
$\mathbb{G}_{a,k}$	$(R,+)$	$k[X]$	$k\{\tau\}$
Nil	$(\mathrm{Nil}(R),+)$	$k[[X]]$	$k[[\tau]]$
$\mathbb{G}^b_{a,k}$	$\varprojlim_\tau \mathrm{Nil}(R)$	$k[[X^{1/q^\infty}]]$	$k((\tau))$

$k[[\tau]]$ is a top. ring (non-comm.) with $\tau^n k[[\tau]]$ basis of neighb. of 0.

$k((\tau)) = \{\sum_{i\geq N} a_i \tau^i : \tau \cdot a = a^q \tau\}$, $N \in \mathbb{Z}$, is a top. field.

Def. $\mathcal{V} : k\text{-Alg} \longrightarrow \mathrm{TopVect}_{\mathbb{F}_q}$ is a <u>formal $\mathbb{F}_q$-vector space</u> if $\mathcal{V} \cong (\mathbb{G}^b_{a,k})^{\oplus d}$.

as $\mathrm{End}(\mathbb{G}^b_{a,k}) \cong k((\tau))$ then

Prop. $\exists$ antiequivalence $\{$formal $\mathbb{F}_q$-vec. spaces$\} \longrightarrow \{$finite free $k((\tau))$-modules$\}$

$\mathcal{V} \longmapsto \mathrm{Hom}(\mathcal{V}, \mathbb{G}^b_{a,k})$.

$\mathcal{V}, \mathcal{V}'$ form. $\mathbb{F}_q$. vec. space of dim d, d' then $\mathrm{Hom}(\mathcal{V}, \mathcal{V}') \cong M_{d',d}(k((\tau)))$

$K = k((\pi))$

Def. A <u>K-formal vector space</u> on k is $\mathcal{V} : k\text{-Alg} \longrightarrow \mathrm{TopVect}_K$ s.t. after composition with $\mathrm{TopVect}_K \to \mathrm{TopVect}_{\mathbb{F}_q}$, $\mathcal{V}$ is a formal $\mathbb{F}_q$-vec. space.

<u>Remark</u>: $\mathcal{V}$ is an $\mathbb{F}_q$-vec. space; $K \longrightarrow \mathrm{End}_{\mathbb{F}_q}(\mathcal{V})$

In the case $d = 1$, such a data is equivalent to $\underset{\mathbb{F}_q((\pi))}{K} \longrightarrow k((\tau)) = \mathrm{End}(\mathbb{G}^b_{a,k})$

$\pi \longmapsto a_n \tau^n + \dots$ $(n > 0)$ n = <u>height of $\mathcal{V}$</u>

Def. K-isocrystal over k: $k((\pi))$-vector space M and $F : M \to M$ automorphism which is $1 \otimes \tau$-semilinear:

$$F\left(\left(\textstyle\sum a_i \pi^i\right) m\right) = \sum a_i^q \pi^i F(m)$$

if $\mathcal{V} \longmapsto M(\mathcal{V}) = \mathrm{Hom}(\mathcal{V}, \mathbb{G}^b_{a,k})$ you obtain an antiequivalence between formal K-vec. spaces and K-isocrystals where F is nilpotent.

Prop. If $\mathcal{V}$ is a formal K-vec. space on k of height n, there exists $\partial : \mathcal{V}^n \longrightarrow \Lambda\mathcal{V}$ which is universal for n-alternating maps.

If A is a local $\mathcal{O}_K$-alg., with ~~the~~ max ideal I s.t. $A/I = k$, $\mathcal{V}_0$ formal K-vec. space on k. Then define:

$$\mathcal{V}: \mathrm{Nilp}_A \longrightarrow \mathrm{TopVect}_K \qquad \text{where } \mathrm{Nilp}_A = A\text{-alg s.t. } I \text{ is nilpotent.}$$
$$R \longmapsto \mathcal{V}_0(A/I)$$

Prop. $\mathcal{V}$ is pro-represented by $A[[X_1^{1/q^\infty}, \dots, X_n^{1/q^\infty}]]$ where $n = $ height $\mathcal{V}_0$.

③ Universal cover

A as above, $A[[X_1, \dots, X_n]]$ the formal additive group equipped with an $\mathcal{O}_K$-module structure. (it is a formal $\mathcal{O}_K$-mod. over A). This defines

$$G: A\text{-Alg} \longrightarrow \mathcal{O}_K\text{-modules}$$
$$R \longmapsto \mathrm{Nil}(R)^n$$

Def. The universal cover $\tilde{G}: A\text{-Alg} \longrightarrow \mathrm{TopVect}_K$

$$R \longmapsto \varprojlim_{\pi} G(R) \quad \pi \text{ acts through } \mathcal{O}_K\text{-mod. str.}$$

If $A = k$, then $\tilde{G}$ is a formal K-vector space.

In general: $\varprojlim_n G(R[[\pi]]) \otimes_{\mathcal{O}_K} K \hookrightarrow \tilde{G}(R)$.

if π is nilpotent in R then this is an isomorphism.

Prop. If A is an $\mathcal{O}_K$-algebra as above, $I \supset \pi A$. Then:

(1) $\tilde{G}(R) \cong \tilde{G}(R/I) \quad \forall_{R \in \mathrm{Nilp}_A}$

(2) if $A/I = k$, then $\tilde{G}$ is a formal K-vect. space.