Local system X smooth adic space $/\mathrm{Spa}(k, O_k)$ k CDV. $k \to \kappa$ res. f.

(1.) $\mathbb{B}_{dR}^+$ - local systems on $X_{proét}$ ($\mathbb{B}_{dR}^+, O\mathbb{B}_{dR}^+, \dots$). sheaves of $\mathbb{B}_{dR}^+$-modules $\mathcal{M}$ locally free of finite rank

we have Poicaré lemma: filtered by t as in the p-adic Hodge theory

$$0 \to \mathbb{B}_{dR}^+ \to O\mathbb{B}_{dR}^+ \xrightarrow{\nabla} O\mathbb{B}_{dR}^+ \otimes \Omega^1 \to \dots$$

on crystalline site, we did: crystal $\mathcal{L} \rightsquigarrow$ resolve $\mathcal{L}$ by cyclic crystals $\rightsquigarrow$ evaluate everything on A_{cr}^+

(2.) étale $\mathbb{Z}_p$-smooth sheaves: definition as usual.

(3.) $O\mathbb{B}_{dR}^+$-modules with connection: - $\mathcal{M}$ loc. free of f. type

- $\nabla_{\mathcal{M}} : \mathcal{M} \to \mathcal{M} \otimes \Omega_X^1$ (over $\nabla_{O\mathbb{B}_{dR}^+}$)

(4.) $\mathcal{E}$ - filtered O_X-modules + integrable connection ∇ (sat. Griffith transversality) (by direct summands)

Properties: (1.) (1) and (3) are equivalent:

$$(1) \ni M \longmapsto M \otimes_{\mathbb{B}_{dR}^+} O\mathbb{B}_{dR}^+ \in (3)$$

$$(3) \ni \mathcal{M} \longmapsto M = \mathcal{M}^{\nabla_{\mathcal{M}}=0} \in (1)$$ M is of right rank over poly. alg.

(2.) $$\{O_{X_{an}}\text{- vector bundles}\} \xrightarrow[eq.]{\sim} \{O_{X_{et}}\text{-v.b.}\} \xrightarrow[eq.]{\sim} \{O_X\text{-v.b. on } X_{proét}\}$$

by faithfully flat descent

$\mathcal{E}$ coh. on $X_{proét}$ becomes v.b. on $U \in X_{proét}$

take $U = \varprojlim U_i$, $U_i \xrightarrow{\text{fin. et.}} U_0 \subset X$ étale open, $U_i \leftrightarrow R_i$ coordinate

then $O_X(U) = \varinjlim R_i = R_\infty$ flat over $R_0 = R$.

and use again f. flat descent.

(3.) $\mathcal{M}$ - $O\mathbb{B}_{dR}^+$ v.b.; $\mathcal{E}$ filtered... Then $\mathcal{E}$ and $\mathcal{M}$ are associated if $\exists$

$$\mathcal{M} \otimes_{O\mathbb{B}_{dR}^+} O\mathbb{B}_{dR} \cong \mathcal{E} \otimes_{O_X} O\mathbb{B}_{dR} \qquad \boxed{\mathcal{E} \longleftrightarrow \mathcal{M}}$$

compatible with filtrations (F^i) and ∇.

(a) if $\mathcal{E} \leftrightarrow \mathcal{M}$ then $M = \mathcal{M}^{\nabla_{\mathcal{M}}=0} = (\mathcal{E} \otimes_{O_X} O\mathbb{B}_{dR})^{F^0, \nabla=0}$

$$\rightsquigarrow \mathcal{E}_{et} \cong \nu_*(M \otimes_{\mathbb{B}_{dR}^+} O\mathbb{B}_{dR})$$ where $\nu : X_{proét} \to X_{et}$.

(b.) $\{\mathcal{E} \text{ filt}\dots\} \xhookrightarrow{\text{f.ll. faith}} \{\mathbb{B}_{dR}^+\text{-local systems}\}$ ← results from a fact for trivial coeffs similar we have proved before.

$$(\mathcal{E}, \nabla, F^\bullet) \longmapsto \mathcal{M}_0 := \mathcal{E} \otimes_{O_X} O\mathbb{B}_{dR}^+$$

$$\rightsquigarrow M_0 = \mathcal{M}_0^{\nabla_{\mathcal{M}_0}=0, F^0} \Rightarrow \mathcal{E} \otimes_{O_X} O\mathbb{B}_{dR}^+ \cong M_0 \otimes_{\mathbb{B}_{dR}^+} O\mathbb{B}_{dR}^+.$$

$\boxed{\mathrm{Coh}.\ \mathcal{E} \leftrightarrow \mathrm{Coh}\ \mathcal{M}}$:

Thm. 7.11: X proper, smooth adic space $/\mathrm{Spa}(k, \mathcal{O}_k)$. $(\mathcal{E}, \nabla, F^\bullet)$ filtered, int. connection ($\rightsquigarrow \mathcal{M}$ $\mathbb{B}_{dR}^+$-local system)

(a) $\exists\ H^i(X_{\bar k}, \mathcal{M}) \otimes_{B_{dR}^+} B_{dR} \overset{\alpha}{\underset{\cong}{\longleftarrow}} H^i_{dR}(X, \mathcal{E}) \otimes_k B_{dR}$

comp. with $\mathrm{Gal}(\bar k/k)$ and $F^\bullet$.

(b) $\exists\ H^i(X_{\bar k}, \mathrm{gr}^0 \mathcal{M}) \cong \bigoplus_j H^{i-j,j}_{\mathrm{Hodge}}(X, \mathcal{E}) \otimes_k \hat{\bar k}(-j)$

comp. with $\mathrm{Gal}(\bar k/k)$ $\qquad = H^i(X, \mathrm{gr}^j(DR(\mathcal{E})))$

proof: $DR(\mathcal{E}) := (0 \to \mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes \Omega^1_X \to \dots)$

$F^m DR(\mathcal{E}) = (0 \to F^m \mathcal{E} \xrightarrow{\nabla} F^{m-1}\mathcal{E} \otimes \Omega^1_X \to \dots)$

To define α: $DR(\mathcal{E}) \longrightarrow DR(\mathcal{E}) \otimes \mathcal{O}\mathbb{B}_{dR}$

$$= \quad 0 \to \mathcal{E} \otimes \mathcal{O}\mathbb{B}_{dR} \to (\mathcal{E} \otimes_{\mathcal{O}} \mathcal{O}\mathbb{B}_{dR}) \otimes \Omega^1_X \to \dots$$

$$\mathcal{M} \otimes_{\mathbb{B}_{dR}^+} \mathbb{B}_{dR}^+ \cong \quad 0 \to \mathcal{M} \otimes_{\mathbb{B}^+_{dR}} \mathcal{O}\mathbb{B}_{dR} \to (\mathcal{M} \otimes_{\mathbb{B}^+_{dR}} \mathcal{O}\mathbb{B}_{dR}) \otimes \Omega^1_X \to \dots$$

Poincaré lemma

claim: α is a filtered q. iso. Reduce to coherent cohomology on gradings.

Lemma: $R\Gamma(X_{\bar k}, \mathcal{A}) \otimes_{\bar k} \mathrm{gr}^i B_{dR} \longrightarrow R\Gamma(X_{\bar k}, \mathcal{A} \otimes \mathrm{gr}^i \mathcal{O}\mathbb{B}_{dR})$.

proof: can assume $i=0$ hence cohom is 0 in dim $j>0$ and $\mathcal{A}$ is $\hat{\bar k}$ in $j=0$. $\square$

Application: (de Rham comparison)

Thm. X proper, smooth $/\mathrm{Spa}(k, \mathcal{O}_k)$; $\mathbb{L}$ $\mathbb{Z}_p$-smooth sheaf $\rightsquigarrow \mathcal{M} = \hat{\mathbb{L}} \otimes_{\mathbb{Z}} \mathbb{B}_{dR}^+$

$\mathrm{Gal}(\bar k/k)$-comp.

$\exists\ H^i(X_{\bar k}, \mathbb{L}) \otimes_{\mathbb{Z}_p} B_{dR}^+ \cong H^i(X_{\bar k}, \mathcal{M})$

If $\mathbb{L}$ is de Rham ($\overset{\text{i.e.}}{\leftrightarrow} (\mathcal{E}, \nabla, F^\bullet)$) then (a) Hodge-de Rham s.seq. degenerates

(b) $H^i(X_{\bar k}, \mathbb{L}) \otimes_{\mathbb{Z}_p} B_{dR} \underset{F^\bullet, G_k}{\cong} H^i_{dR}(X, \mathcal{E}) \otimes_k B_{dR}$.

proof: have $H^i(X_{\bar k}, \mathbb{L}_n) \otimes_{\mathbb{Z}_p} A_{inf}^a \cong H^i(X_{\bar k}, \mathbb{L}_n \otimes_{\mathbb{Z}_p} \mathbb{A}^a_{inf})$ $\leftarrow$ dévissage and gradings (Thm 5.1)

pass to $\varprojlim$ over n and invert t:

$$H^i(X_{\bar k}, \mathbb{L}) \otimes_{\mathbb{Z}_p} (B^a_{inf})/(\ker\theta)^m \cong H^i(X_{\bar k}, \hat{\mathbb{L}} \otimes_{\mathbb{Z}_p} (\mathbb{B}^a_{inf})/(\ker\theta)^m)$$

pass to the $\varprojlim_m$ $\square$