GDT A. Schenabish - "FF curve"

p-adic Hodge theory $\rightsquigarrow$ p-adic Galois representations of $G_{\mathbb{Q}_p}$.

70's: B_{dR}, B_{cris} (Fontaine), study of modules over these rings.

2011: Fargues - Fontaine: study Gal. reps by studying vector bundles over a curve.

motivation: 65's Narasimhan - Seshadri: indecomposable v.b. over cpt Riemann surface are stable $\Leftrightarrow$ they come from irr. proj. unitary reps of π_1.

① Simplest possible description

C alg. closed, char 0 complete w.r.t. non-triv. non-arch. field and res. char p.

$C \rightsquigarrow B_{dR}, B_{st}, B_{cris}$ (Fontaine theory) ; $B_e = (B_{cris})^{\varphi=1}$.

Thm: $(B_e)^\times = \mathbb{Q}_p^\times$. B_e is PID. [Kedlaya, Berger: B_e is a Bezout domain]

Take $X^e = \mathrm{Spec}(B_e)$, $X = X^e \amalg \{\infty\}$; $B_e \subseteq B_{dR}$, v_{dR} discrete valuation on B_{dR}

$C_e = \mathrm{Frac}(B_e)$, $\mathcal{O}_{X,\infty} = \{b \in C_e : v_{dR}(b) \geq 0\}$ so actually we put $X = X^e \amalg_{\mathrm{Spec}\, C_e} \mathcal{O}_{X,\infty}$.

$$\Gamma(U, \mathcal{O}_X) = \begin{cases} \Gamma(U, \mathcal{O}_{X^e}) & \text{if } \infty \notin U \\ \Gamma(U \setminus \{\infty\}, \mathcal{O}_{X^e}) \cap B_{dR}^+ & \text{if } \infty \in U \end{cases}$$

X is separated, integral Noetherian normal scheme, regular but not smooth (not finite type).
($\Rightarrow \mathrm{Pic}(X \setminus \{\infty\}) = 0$ and $\deg: \mathrm{Pic}(X) \xrightarrow{\sim} \mathbb{Z}$ because of B_e is PID.

vector bundles and their cohomology

$\mathcal{F} = (F_e, \hat{F}_\infty)$ is a vector bundle on X, where F_e = free B_e-module of f. rank

(B-pairs), Berger '07) $\rightsquigarrow$ (φ, Γ)-mod / Robba ring $\rightsquigarrow$ Gal. reps

$\hat{F}_\infty = B_{dR}^+ \otimes_{\mathcal{O}_{X,\infty}} F_\infty$ is a B_{dR}-lattice inside $F_{dR} = B_{dR} \otimes_{B_e} F_e$.

we have:

$$(*) \quad 0 \to H^0(X, \mathcal{F}) \to \mathcal{F}_e \oplus \hat{\mathcal{F}}_\infty \to \mathcal{F}_{dR} \to H^1(X, \mathcal{F}) \to 0.$$

recall fund. sequence: $0 \to \mathbb{Q}_p = B_e \cap B_{dR}^+ \to B_e \oplus B_{dR}^+ \to B_{dR} \to 0$

$\qquad \mathrm{Fil}^0(B_{cris})^{\varphi=1}$ (this is $(*)$ for $\mathcal{F} = \mathcal{O}_X$)

Cor.: $H^0(X, \mathcal{O}_X) = \mathbb{Q}_p$, $H^1(X, \mathcal{O}_X) = 0$ ("genus" $= 0$)

remark. B_e is "almost Euclidean" $\Rightarrow H^1(X, \mathcal{O}_X(k)) = 0 \ \forall k \geq 0$
But $H^1(X, \mathcal{O}_X(-k)) \neq 0$ and in fact are infinite-dim spaces.

We want to study semi-stable v.b. of slope 0 (they correspond to Gal. reps)

$$\text{slope} = \frac{\text{degree}}{\text{rank}}$$

~~[illegible]~~ 2. Another description of the curve: (tie up with perfectoid fields)

F = alg. closed field of char p, complete w.r.t. non-trivial non-arch. abs. value $|\cdot|$

E = locally cpt non-arch. field with res. field $\mathbb{F}_q \subseteq F$, π uniformiser. $((\mathcal{O}_{\mathbb{C}_p}/p)^\flat = R)$

want to construct $X = X(E, F, \pi)$; previous description was for $X(F(C), \mathbb{Q}_p, p)$.

$\mathcal{E}$ = unique field $\supseteq E$ complete w.r.t. with res. field F.

(equal char.): $E = \mathbb{F}_q((\pi))$, $\mathcal{E} = F((\pi))$, (mixed char.): $E/\mathbb{Q}_p < \infty$, $\mathcal{E} = E \otimes_{W(\mathbb{F}_q)} W(F)$.

we have a map $F \to \mathcal{O}_{\mathcal{E}}$, $a \mapsto [a]$ $\quad \mathcal{E} = \{ \sum_{n \gg -\infty} [a_n]\pi^n \mid a_n \in F \}$

$B^b = \{ \sum_{n \gg -\infty} [a_n]\pi^n \mid \exists_c \ |a_n| \le c \ \forall_n \}$

if $f \in B^b$, $\rho \in [0,1]$, $|f|_\rho = \begin{cases} q^{-r}, & r = \text{smallest integer s.t. } a_r \neq 0, \text{ if } \rho = 0 \\ \sup |a_n|\rho^n, & \text{if } \rho \neq 0 \end{cases}$

Remark: sup = max when $\rho < 1$, because $\to 0$; Fact: $|\cdot|_\rho$ is a mult. norm.

B_I = completion of B^b w.r.t. ~~[illegible]~~ $|\cdot|_\rho$ for $\rho \in I$.

$B = B_{(0,1)} = \varprojlim_{I \in (0,1) \text{ closed}} B_I$; elts in B_I are "rigid-analytic functions converging on ~~[illegible]~~ $\in I$".

$B_I = \{ f = \sum_{n=-\infty}^{\infty} [a_n]\pi^n \ \ \forall_{\rho \in I} |a_n|\rho^n \to 0$ as $n \to \pm\infty \}$ remark: it's not clear if every elt $f \in B_I$ can be written in this form.

if $I \subset J$ then $B_J \hookrightarrow B_I$. Define $Y_I = \operatorname{Spec} B_I$, $Y = \varinjlim_I Y_I$

φ = Frob acts on B^b via: $\sum [a_n]\pi^n \mapsto \sum [a_n^q]\pi^n$. $\rightsquigarrow \varphi$ acts on B_I $\rightsquigarrow$ on B $\rightsquigarrow$ on Y.

Define: $X = Y/\varphi^{\mathbb{Z}}$.

3. Explicit description: $X = \operatorname{Proj} P$ where $P = \bigoplus P_d$ where $P_d = \{ b \in B : \varphi(b) = \pi^d b \}$.

so we have 3 descriptions: (I): $X = \operatorname{Spec} B_e \amalg \{\infty\}$ (II) $X = Y/\varphi^{\mathbb{Z}}$ (III) $X = \operatorname{Proj} P$.

Prop. (1) $\lambda \in F$, $|\lambda| \in I$, ideal of B_I gen. by $(\pi - [\lambda])$ is maximal

(2) I is closed, B_I is PID. If $\mathfrak{p}$ is max. $\exists\, \lambda \in F$ with $|\lambda| \in I$ s.t. $\mathfrak{p} = (\pi - [\lambda])$.

in equal char. such a λ is unique and $B_I/\mathfrak{p} = F$.

in mixed char. λ is not unique, $B_I/\mathfrak{p} = C_p$ (non-arch. alg. closed ...); $|\lambda|$ is unique

$|\pi|_p = |\lambda|$, $F = F(C_p)$

$\theta : B_I \to C_p$ extend. $B^b \to C_p$ $\quad (\sum [a_n]\pi^n \mapsto \sum a_n^{(0)} \pi^n)$

Comments on λ not unique. $\lambda^{(0)} = \pi$, $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots)$. For every λ, $\exists$ a $\mathbb{Z}_p$-line inside the multiplicative gp of units of F (depending on λ).

It is hard to determine λ.

The ring P. $E = \mathbb{Q}_p$, $\pi = p \rightsquigarrow P_0 = \mathbb{Q}_p$, put $U = 1 + m_F$ ($\mathbb{Q}_p$-Banach space).

$$1 + m_F \xrightarrow{\sim} P_1 \quad \circlearrowleft \varphi \;;\; P \text{ is a graded factorial ring i.e. } x \in P_d \Rightarrow \exists_{t_1, \dots, t_d \in P_1} \text{ s.t. } x = t_1 \cdots t_d$$

$$u \longmapsto \log [u]$$

Sketch on proof: (x) ideal generated in B_I. Step 1: $(x) \subseteq m = (p - [\lambda])$, $\varphi(x) = p^d x$.

the ideal generated by x is fixed by φ^n $\forall_n$.

Look at $(x) \cap \bigcap_n \varphi^n(m) \in P_1$.

we have to show it!

Now: $m \cap P_1 = \log$ of unique $\mathbb{Z}_p$-line; $\varepsilon \mathbb{Z}_p$ in U $\leftrightarrow$ free $\mathbb{Z}_p$-module of rk 1.

closed points of X (the curve) $\longleftrightarrow$ $\mathbb{Q}_p$-lines in P_1.

[closed max. ideals of B $\Leftrightarrow$ closed max. ideals of B_I].

$\updownarrow$

$Y/\varphi^{\mathbb{Z}}$.

explicit map: $\varepsilon \in m_F \setminus \{0\}$, $\alpha = 1 + \varepsilon$, $u_\varepsilon = 1 + [\alpha] + [\alpha^{1/p}] + \dots + [\alpha^{\frac{p-1}{p}}]$

$\varepsilon \longmapsto P_\varepsilon = (u_\varepsilon)$.

(general) **Thm.** $P = \bigoplus P_d$, $P_0 = K$ (some field), $\dim_K P_1 \geq 2$

Assume the following: ⓐ the mult. monoid $\left(\bigcup_{d \geq 1} P_d \setminus \{0\}\right)/K^\times$ is free over elts of $P_1/K^\times$. ⓑ $t \in P_1 \setminus \{1\}$, $\exists C/K$, $P/tP \cong \{f \in C[T] : f(0) \in F\} =: D$ is a graded K-alg.

Then the following are true: ① $t \in P_1 \setminus \{0\}$, the vanishing locus of the hyperplane section is $V^+(t) = \{\infty_t\}$.

② $|X|$ = closed pts, $t \longmapsto \{\infty_t\}$

$P_1 \setminus \{0\}/K^\times \xrightarrow{\sim} |X|$

($X = \mathrm{Proj}\, P$)

③ deg (closed pt) = 1 $\rightsquigarrow$ X is a complete curve

④ $\infty \in X$, $X \setminus \{\infty\}$ open affine $\mathrm{Spec}(B)$, B PID.

where $B = P[\frac{1}{t}]_{(0)}$ = elts of degree 0.

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take $E = \mathbb{Q}_p$, $\varepsilon \in m_F \setminus \{0\}$, $m = (u_\varepsilon)$, $Q(x) = (1+x)^p - 1$, $t = \log([1+\varepsilon])$

$$\rightsquigarrow 0 \to \mathbb{Q}_p t^m \to (B^+_{cris})^{\varphi = p^d} \to B^+/_{\ker m^d} \to 0$$