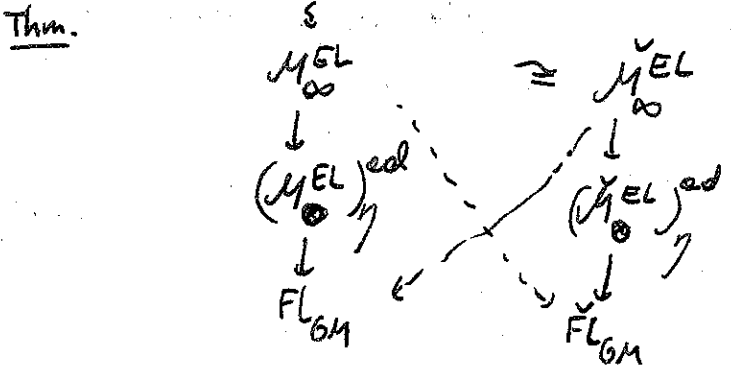


GDT. H. Wang "RZ-species II"

We want to prove duality thm for RZ species of EL structure at  $\infty$ .

We have  $(B, V, \lambda, b, \mu)$  datum  $\xrightarrow{\text{basic}}$  dual  $(\check{B}, \check{V}, \check{\lambda}, \check{b}, \check{\mu})$



Corollary:  $M_{\infty}^{LT} \cong M_{\infty}^{Dr}$   $\xrightarrow{\text{Faltings-Fargues iso}}$   $RM_c^{\Gamma}(M_{\infty}^{LT}) \cong RM_c^{\Gamma}(M_{\infty}^{Dr})$   
 $GL_n \times D^*$  equiv.  $GL_n \times D^*$

§1. EL structures  $k = \bar{F}_q, W(k), \sigma: W(k) \rightarrow W(k)$  Frobenius

$B$  semisimple f.d.  $\mathbb{Q}_p$ -alg,  $B = \prod M_{n_i}(D_i)$   $D_i$  - div. alg.

$\bigcup \mathcal{O}_B$  maximal compact  $\mathcal{O}_B = \prod M_{n_i}(\mathcal{O}_{D_i})$

$V$  finite  $B$ -module,  $\dim_{\mathbb{Q}_p} V = k$ ,  $\lambda \subset V$   $\mathcal{O}_B$ -lattice,  $G = GL_B(V)$  alg. gr. /  $\mathbb{Q}_p$ .

fix  $\mu: G_m \rightarrow G_{\mathbb{Q}_p}$  defined over  $E/\mathbb{Q}_p$  finite.

$V = \bigoplus V_i, V_i = \{v \in V \mid \mu(z)v = z^i v \ \forall z \in E^*\}$  weight decomp.

Assume:  $\mu$  has weights 0 and 1  $\leadsto V = V_0 \oplus V_1$

$b \in G(W(k)[1/p]) / \sim_{\mathcal{O}_B}$  gives an isocrystal  $N_b = (V \otimes W(k)[1/p], b(\text{id} \otimes \sigma))$

Assume  $N_b$  has slopes  $\in [0, 1]$ .  $\exists$   $p$ -div. sp  $H_{1/k}$  with  $\mathcal{O}_B$ -action

$N_b \cong_{\mathcal{O}_B\text{-sp.}} M(H)[1/p]$ . Assume  $(b, \mu)$  is admissible.

Let  $X$   $p$ -div. of  $K, K/W(k)[1/p]$  finite s.t.  $\exists$   $p$ -iso.  $H \otimes_{\mathcal{O}_K/m_K} \rightarrow X \text{ mod } m_K$

and the fibration  $0 \rightarrow (\text{Lie } X^{\vee})^{\vee} \otimes K \rightarrow M(H)[1/p] \rightarrow \text{Lie } X \otimes K \rightarrow 0$

coincides with  $0 \rightarrow V_1 \rightarrow V \rightarrow V/V_1 \rightarrow 0$ .  $H$  is ef wth  $h$  and  $\dim \mathcal{O}$ .

Thm. (RZ.)  $\check{E} = E \cdot W(k)$ . We have  $M^{EL}: \text{Nil}_{p, \mathcal{O}_E} \rightarrow \text{sets}$

which is representable by

a formal scheme  $\text{spf}(\check{E}, \mathcal{O}_E^{\vee})$

which admits a locally f.p. ideal of definition.

$R \mapsto \mathbb{P}_2(G, \rho) \mid \left. \begin{array}{l} G \text{ } p\text{-div. sp. with } \mathcal{O}_B\text{-action} \\ \text{and determinant condition} \end{array} \right\}$   
 $\rho: H \otimes_{\mathbb{Z}} R/p \xrightarrow{\text{e. iso.}} G \otimes_{\mathbb{Z}} R/p$   
 $\uparrow$   
 $\mathcal{O}_B\text{-lin}$

finite level structure:  $\forall n \geq 1 \quad M_n^{EL} = \{ \text{complete aff. } (\check{E}, \mathcal{O}_{\check{E}})\text{-alg} \} \rightarrow \text{Sets}$   
 $(R, R^+) \mapsto \{ (G, \rho, \alpha) \mid (G, \rho) \in (M_n^{EL, ad})(R, R^+) \}$   
 $\alpha: \Lambda/p^n \rightarrow G[\rho^n]_{\gamma}(R, R^+)$   $\mathcal{O}_B$ -linear  
 and  $\forall x = \text{Spa}(K, K^+) \in \text{Spa}(R, R^+)$   $\rho(K)$  is isom.

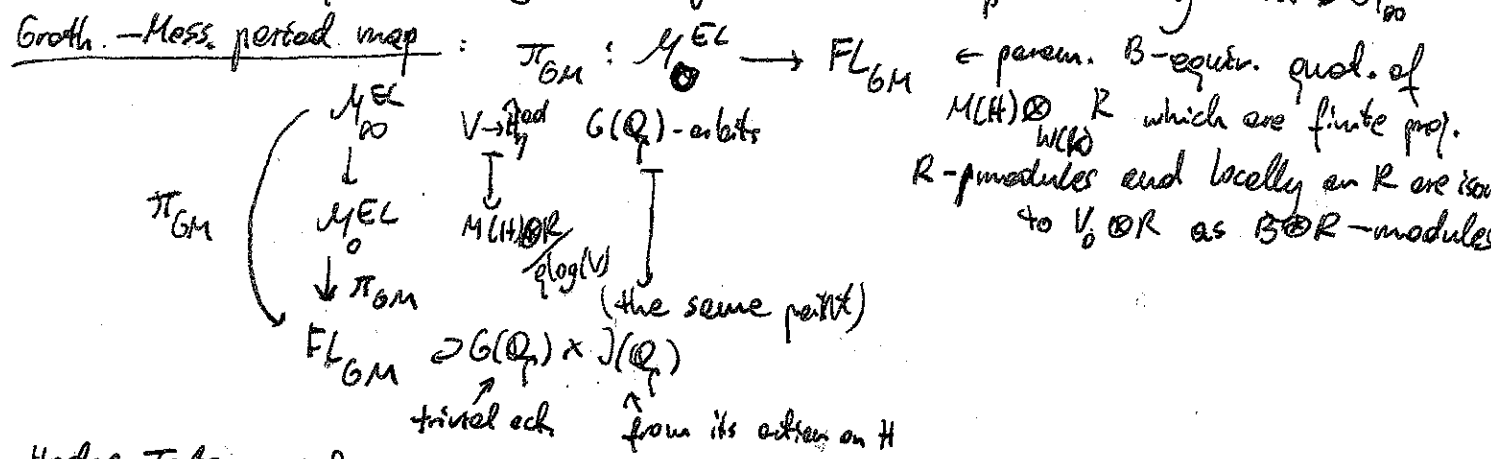
infinite level structure:  $M_{\infty}^{EL}: (R, R^+) \mapsto \{ (G, \rho, \alpha) \mid (G, \rho) \in (M_{\infty}^{EL, ad})(R, R^+) \}$   
 $\alpha: \Lambda \rightarrow T(G)_{\gamma}^{ad}(R, R^+)$   $\mathcal{O}_B$ -lin + iso on  $x = \text{Spa}(K, K^+)$

Thm.  $M_{\infty}^{EL}$  is rep. by an adic space over  $\text{Spa}(\check{E}, \mathcal{O}_{\check{E}})$   
 It is pre-perfectoid,  $M_{\infty}^{EL} \sim \varprojlim_n M_n^{EL}$ . Moreover we have the following description of  $M_{\infty}^{EL}$ : it is a sheafification of:

- $(R, R^+) \mapsto \{ \mathcal{B}\text{-mod. morph. } V \xrightarrow{\alpha} \hat{H}_{\gamma}^{ad}(R, R^+) \text{ s.t.} \}$
- (1)  $V \xrightarrow{\alpha} \hat{H}_{\gamma}^{ad}(R, R^+) \xrightarrow{\rho \circ \log} M(H) \otimes R$ . The prokernel of this morph. is loc. isom. to  $V_0 \otimes R$  over  $R$ . ( $V_0$  corresp. to  $\text{Lie}(G)$ )
  - (2)  $\forall x = \text{Spa}(K, K^+) \in \text{Spa}(R, R^+)$  the map  $V \rightarrow \hat{H}_{\gamma}^{ad}(K, K^+)$  is injective

Group actions on  $M_{\infty}^{EL}$ :  
 •  $G(\mathbb{Q}_p)$  acts on  $M_{\infty}^{EL}$  ( $G = GL_B(V)$ )  
 •  $J(\mathbb{Q}_p) = \text{End}_B(H) = \{ g \in GL_B(V) \mid \sigma g = b^{-1} g b \}$

Remark: (1) that avoids the use of Hecke corr.  
 (2)  $J(\mathbb{Q}_p)$  acts ~~on~~ every  $M_n^{EL}$  and  $G(\mathbb{Q}_p)$  acts only on  $M_{\infty}^{EL}$



Hodge-Tate period map: We will define  $M_{\infty}^{EL} \rightarrow FL_{HT}$   
 (only acts at  $\infty$ -level) let  $FL_{HT}$  be the adic space param.  
 $(R, R^+) \mapsto \mathcal{B}$ -equiv. quotient of  $V \otimes R$  which are loc. isom. to  $V_1 \otimes R$  as  $\mathcal{B} \otimes R$ -mod.

Prop.  $\exists$  HT period map  $\pi_{HT} : M_{\infty}^{EC} \rightarrow FL_{HT} \cong G(Q_p) \times J(Q_p)$  - equiv.  $S_G(Q_p) \times J(Q_p) \leftarrow$  trivial action.

proof:  $V \in M_{\infty}^{EC} \rightsquigarrow p$ -div sp  $(G, \rho) \in (M_{\infty, \eta}^{EC, ad})(R, R^+) \rightsquigarrow V \rightarrow V(G)_{\eta}^{ad}$

$V \xrightarrow{\alpha} V(G)_{\eta}^{ad}(R, R^+) \rightarrow \tilde{H}_{\eta}^{ad}(R, R^+) \rightarrow \text{Lie } G^{\vee} \otimes R \rightarrow (\text{Lie } G^{\vee})^{\vee} \otimes R$

Claim: this map is surjective

proof of claim: enough to check it on points...  $x = \text{Spa}(K, K^+) \in \text{Spa}(R, R^+)$ .

The quot of  $M(H) \otimes K / \mathfrak{q}_{\log(V)} \cong V_0 \Rightarrow \dim(\mathfrak{q}_{\log(V)}) = h-d \Rightarrow$  surjectivity

define:  $\pi_{HT}(x) :=$  quotient  $V \otimes R \rightarrow (\text{Lie}(G^{\vee})^{\vee} \otimes R)$  (loc isom. to  $V_1 \otimes R$ )

§2. The dual data

Fact:  $b \in G(W(W)[\frac{1}{p}]) / \sim$  is basic iff  $\check{B} := \text{End}_{\check{B}}^{\circ}(H) = \text{End}_{\check{B}}(H) \otimes \mathbb{Q}$

$\check{B} \otimes_{\mathbb{Q}_p} W(W)[\frac{1}{p}] = \text{End}_{\check{B} \otimes W(W)}(M(H)[\frac{1}{p}])$ , in particular  $\check{B} = \text{End}_{\check{B}}(V)$

define dual data:  $\check{B} = \text{End}_{\check{B}}^{\circ}(H) = \text{End}_{\check{B}}(V) \supset \mathcal{O}_{\check{B}} = \text{End}_{\mathbb{O}_B}(H)$

let  $\check{V} = \check{B} \supset \check{\Lambda} = \mathcal{O}_{\check{B}}$ .  $\check{G} = GL_{\check{V}}(\check{B}) \cong \mathbb{J} \subset \check{B}$

$\check{H} := \Lambda^* \otimes_{\mathbb{O}_B} H$  (i.e.  $M(\check{H}) = \Lambda^* \otimes_{\mathbb{O}_B} M(H) \supset \mathcal{O}_{\check{B}}$ ),  $\Lambda^* = \text{Hom}_{\mathbb{O}_B}(\Lambda, \mathcal{O}_B)$

and  $\check{\mu} : G_M \rightarrow \check{G}_{\check{B}} \cong \check{G} \otimes W(W)[\frac{1}{p}] \cong \text{End}_{\check{B}}(H) \otimes W(W)[\frac{1}{p}] = \text{End}_{\check{B}}(M(H)[\frac{1}{p}]) = \text{End}_{\check{B}}(M(H)[\frac{1}{p}]) \otimes W(W)[\frac{1}{p}] = \text{End}_{\check{B}}(V \otimes W(W)[\frac{1}{p}]) = \text{Hom}_{\check{B}}(V_0, V) \otimes W(W)[\frac{1}{p}] \oplus \dots = \check{V}_0$

define  $\check{\mu}$  via:  $\text{End}_{\check{B}}(V \otimes W(W)[\frac{1}{p}]) = (\text{Hom}_{\check{B}}(V_0, V) \otimes W(W)[\frac{1}{p}]) \oplus \dots = \check{V}_0$

Prop. There is a natural action of  $G$  on  $\check{V}$  and  $FL_{GM}$  and  $FL_{HT} \cong FL_{GM}$

proof: Recall:  $FL_{HT} \rightsquigarrow \mathbb{B}$ -equiv. quot. of  $V \otimes R$  finite. prof.  $R$ -mod. loc. isom. to  $V_1 \otimes R$  as  $\mathbb{B} \otimes R$  modules.

points of  $FL_{GM} \rightarrow (V \otimes R \rightarrow W_1 \cong_{loc} V_1 \otimes R) \rightarrow \check{B}$ -equiv. quot. of  $M(\check{H}) \otimes R$  loc. isom. to  $\text{Hom}_{\check{B}}(V_0, V) \otimes R$

s.t.  $\text{End}_{\check{B}}(V) \otimes R \rightarrow W \cong_{loc} \text{Hom}_{\check{B}}(V_0, V) \otimes R$

so now we get the duality by:  $0 \rightarrow W_0 \xrightarrow{\cong V_0} V \otimes R \rightarrow W_1 \rightarrow 0$   $\cong V_1 \otimes R$   
 apply  $\text{Hom}_B(-, V) \rightarrow 0 \rightarrow \text{Hom}_B(W_1, V) \rightarrow \text{End}_B(V) \otimes R \rightarrow \text{Hom}_B(W_0, V) \rightarrow 0$

Claim. this is an isom. of flag varieties.

The action of  $G$  on  $FL_{HT}$  identifies with an action on  $\check{F}l_{GM}$ .

similar argument for  $FL_{GM} \cong \check{F}l_{HT}$   $\text{Hom}_B(V_0, V)$

Thm. (Scholze-Weinstein)

$\exists G(\mathbb{Q}_p) \times \check{G}(\mathbb{Q}_p)$  - equiv. isom.

$$\begin{array}{ccc} M_{\infty}^{EL} & \cong & \check{M}_{\infty}^{EL} \\ \pi_{GM} \downarrow & & \downarrow \check{\pi}_{HT} \\ FL_{GM} & \cong & FL_{HT} \end{array} \quad \text{identifying } \pi_{GM} \text{ with } \check{\pi}_{HT}$$

proof:  $(R, R^+)$  complete aff.  $(\check{E}, \check{O}_{\check{E}})$ -alg. Let  $s \in M_{\infty}^{EL} \leftrightarrow V \rightarrow \check{H}_{\check{y}}^{\text{ad}}(R, R^+)$  s.t. the quotient  $M(H) \otimes R \rightarrow W \cong_{\text{loc}} V_0 \otimes R$  and is injective at every (geom.) point.  
 $\check{s} \in \check{M}_{\infty}^{EL} \leftrightarrow \check{V} \rightarrow \check{H}_{\check{y}}^{\text{ad}}(R, R^+)$  s.t. the quotient  $M(\check{H}) \otimes R \rightarrow \check{W} \cong_{\text{loc}} \text{Hom}_B(V_0, V) \otimes R$  and is injective at every (geom.) point.

we identify  $\check{s}: \check{V} = \text{End}_B(H) \left[ \begin{smallmatrix} 1 \\ \rho \end{smallmatrix} \right] \rightarrow V^* \otimes \check{H}_{\check{y}}^{\text{ad}}(R, R^+)$

$$\begin{array}{ccc} \text{Start with } s: V \rightarrow \check{H}_{\check{y}}^{\text{ad}}(R, R^+) & \mapsto & \check{s}: \text{End}_B(H) \rightarrow V^* \otimes \check{H}_{\check{y}}^{\text{ad}}(R, R^+) \\ \downarrow f \in \text{End}_B(H) & \rightsquigarrow & \downarrow \check{f} \\ \check{H}_{\check{y}}^{\text{ad}}(R, R^+) & & \check{H}_{\check{y}}^{\text{ad}}(R, R^+) \end{array} \quad \check{f} \mapsto (v \mapsto \check{f}(s(v)))$$

claim:  $\check{s} \mapsto s(v) = \check{s}(\text{id})(v)$ .

This gives a bijection

For the quotient: claim: the map induced by  $\check{s}: \check{V} \otimes R \rightarrow M(\check{H}) \otimes R$

is induced by  $\text{Hom}_B(-, V)$  applied to the  $\text{End}_B(H) \otimes R \rightarrow \text{End}_B(V)$

map induced by  $s: V \otimes R \rightarrow M(H) \otimes R \rightsquigarrow V \otimes R \rightarrow (\text{Lie } G^v)^v \otimes R$

and this is compatible with period maps.

□

Example: LT:  $B = \mathbb{Q}_p$ ,  $H$  formal  $\mathbb{Z}_p$ -module of dim 1 and ht  $n$ .

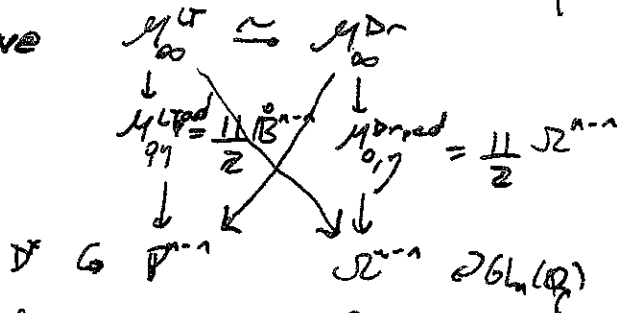
$V = \mathbb{Q}_p^n$ ,  $G = GL_n(\mathbb{Q}_p)$ ,  $J = (D_{\mathbb{Q}_p}^{1/n})^*$ .  $Fl_{GM} = \mathbb{P}^{n-1}$ ,  $\mathcal{M}_{\infty}^{LT}$

Drinfeld:  $\check{B} = D_{\mathbb{Q}_p}^{1/n} = \check{V}$ ,  $\check{G} = J$   $\dim \check{V}_0 = n$ ,  $\dim \check{V}_1 = n^2 - n$

$\check{H}$  = special formal  $\mathbb{Q}_p$ -module of dim  $n$ , ht  $n^2$ .

$Fl_{GM} = \Omega^{n-1} = \mathbb{P}^{n-1} \setminus \cup \mathbb{Q}_p$ -vert. hyperplanes,  $\mathcal{M}_{\infty}^{DR}$ .

We have



Application: Thm. Let  $\tilde{\Omega} \rightarrow \Omega$  be finite étale  $GL_n(\mathbb{Q}_p)$ -equivariant.

Then  $\exists_n$  s.t.  $\mathcal{M}_{\infty}^{DR} \xrightarrow{\dots} \tilde{\Omega}$

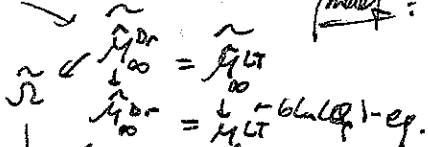
Proof:  $\hat{\mathcal{M}}_{\infty}^{DR} \cong \hat{\mathcal{M}}_{\infty}^{LT} / GL_n(\mathbb{Q}_p) \times \mathbb{Z}$

Lemma:  $\{ \text{set of fin. étale } n\text{-adic spaces } / \mathbb{P}^{n-1} \} \cong$

$\cong \{ GL_n(\mathbb{Q}_p)\text{-equiv. finite étale } / \mathbb{P}^{n-1} \}$

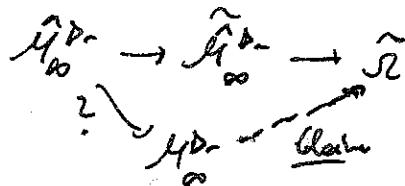
proof:  $\hat{\mathcal{M}}_{\infty}^{LT} / GL_n(\mathbb{Q}_p) \cong \mathbb{P}^{n-1}$

fiber product



We have

By rigid GAGA, any finite étale map over  $\mathbb{P}^{n-1}$  is algebraic and global hence



have to prove the claim.

□