GDT. H. Wang "RZ spaces II"

①

We want to prove duality thm for RZ spaces of EL structure at ∞.

We have $(B, V, \Lambda, b, \overset{\text{basic}}{\mu})$ datum $\overset{\text{dual}}{\rightsquigarrow}$ $(\check{B}, \check{V}, \check{\Lambda}, \check{b}, \check{\mu})$

Thm.

$$\begin{tikzcd}
\mathcal{M}_\infty^{EL} \arrow[d] \arrow[rr, phantom, "\cong"] \arrow[ddrr, dashed] & & \check{\mathcal{M}}_\infty^{EL} \arrow[d] \arrow[ddll, dashed] \\
(\mathcal{M}_{\circledast}^{EL})^{ad}_\eta \arrow[d] & & (\check{\mathcal{M}}_{\circledast}^{EL})^{ad}_\eta \arrow[d] \\
\mathcal{F}\ell_{GM} & & \check{\mathcal{F}\ell}_{GM}
\end{tikzcd}$$

<u>Corollary</u>: $\underset{GL_n \times D^\times \text{ equiv.}}{\mathcal{M}_\infty^{LT} \cong \mathcal{M}_\infty^{Dr}}$ $\rightsquigarrow$ Faltings – Fargues iso $R\Gamma_c(\mathcal{M}_\infty^{LT}) \underset{GL_n \times D^\times}{\cong} R\Gamma_c(\mathcal{M}_\infty^{Dr})$

§1. EL structures

$k = \overline{\mathbb{F}}_p$, $W(k)$, $\sigma : W(k) \to W(k)$ Frobenius

B semisimple f.d. $\mathbb{Q}_p$-alg, $B = \prod_i M_{n_i}(D_i)$ D_i - div. alg.

$\overset{\cup}{\mathcal{O}_B}$ maximal compact $\qquad \overset{\cup}{\mathcal{O}_B} = \prod_i M_{n_i}(\mathcal{O}_{D_i})$

V finite B-module, $\dim_{\mathbb{Q}_p} V = h$, $\Lambda \subset V$ $\mathcal{O}_B$-lattice, $G = GL_B(V)$ alg. gr. $/\mathbb{Q}_p$.

fix $\mu : \mathbb{G}_m \longrightarrow G_{\overline{\mathbb{Q}}_p}$ defined over $E/\mathbb{Q}_p$ finite.

$V = \bigoplus_i V_i$, $V_i = \{ v \in V \mid \mu(z)v = z^i v \quad \forall z \in E^\times \}$ weight decomp.

<u>Assume</u>: μ has weights 0 and 1 $\rightsquigarrow$ $\underset{\mathbb{Q}_p}{V} = V_0 \oplus V_1$.

$b \in G(W(k)[1/p])/\sim_\sigma$ gives an isocrystal $N_b = (V \otimes_{\mathbb{Q}_p} W(k)[\frac{1}{p}], b(\mathrm{id} \otimes \sigma))$ $\overset{B}{\circlearrowleft}$.

Assume N_b has slopes $\in [0, 1]$. $\exists$ p-div gp $H/_k$ with $\mathcal{O}_B$-action $N_b \underset{\mathcal{O}_B\text{-eq.}}{\cong} M(H)[\frac{1}{p}]$. Assume (b, μ) is admissible.

Let X p-div$_{\mathcal{O}_K}$, $K/W(k)[\frac{1}{p}]$ finite s.t. $\exists_{q.\text{iso.}}$ $H \otimes \mathcal{O}_K/m_K \longrightarrow X \bmod m_K$

and the filtration $0 \to (\mathrm{Lie}\, X^\vee)^\vee \otimes K \to M(H)[\frac{1}{p}] \to \mathrm{Lie}\, X \otimes K \to 0$

coincides with $0 \to V_1 \to V \to V/V_1 \to 0$. $\quad H$ is of ht h ~~and~~ and dim d.

<u>Thm.</u> (R.Z.) $\breve{E} = E \cdot W(k)$. We have $\mathcal{M}^{EL} : \mathrm{Nilp}_{\mathcal{O}_{\breve{E}}} \longrightarrow$ Sets

$R \longmapsto \left\{ (G, \rho) \;\middle|\; \begin{array}{l} G \text{ } p\text{-div. gp}/_R \text{ with } \mathcal{O}_B\text{-action} \\ \text{and determinant condition} \\ \rho : H \otimes_k R/p \xrightarrow{q.\text{isog.}} G \otimes_R R/p \\ \qquad\qquad\quad \uparrow \\ \qquad\qquad \mathcal{O}_B\text{-lin} \end{array} \right\} /\cong$

which is representable by a formal scheme $/\mathrm{Spf}(\breve{E}, \mathcal{O}_{\breve{E}})$ which admits a locally f.g. ideal of definition finite level structure: $\forall_{n\geq 1}$ $\mathcal{M}_n^{EL}$: {complete aff. $(\breve{E}, \mathcal{O}_{\breve{E}})$-alg} $\mapsto$ Sets

$$(R,R^+) \longmapsto \left\{ (G,\rho,\alpha) \;\middle|\; \begin{array}{l} (G,\rho) \in (\mathcal{M}_\eta^{EL,ad})(R,R^+) \\ \alpha : \Lambda/p^n \longrightarrow G[p^n]_\eta^{ad}(R,R^+) \;\; \mathcal{O}_B\text{-linear} \\ \text{and } \forall_{x=\mathrm{Spa}(K,K^+)\in \mathrm{Spa}(R,R^+)} \;\; \alpha(x) \text{ is isom.} \end{array} \right\}$$

infinite level structure: $\mathcal{M}_\infty^{EL} : (R,R^+) \mapsto \left\{ (G,\rho,\alpha) \;\middle|\; \begin{array}{l} (G,\rho) \in (\mathcal{M}_\eta^{EL,ad})(R,R^+) \\ \alpha : \Lambda \to T(G)_\eta^{ad}(R,R^+) \;\; \mathcal{O}_B\text{-lin} + \text{iso on } x=\mathrm{Spa}(K,K^+) \end{array} \right\}$

Thm. $\mathcal{M}_\infty^{EL}$ is rep. by an adic space over $\mathrm{Spa}(\breve{E}, \mathcal{O}_{\breve{E}})$

It is pre-perfectoid, $\mathcal{M}_\infty^{EL} \sim \varprojlim_n \mathcal{M}_n^{EL}$. Moreover we have the following description of $\mathcal{M}_\infty^{EL}$: it is a sheafification of:

$(R,R^+) \longmapsto$ {B-mod. morph : $V \xrightarrow{\alpha} \tilde{H}_\eta^{ad}(R,R^+)$ s.t.

(1) $V \xrightarrow{\alpha} \tilde{H}_\eta^{ad}(R,R^+) \xrightarrow{q\log} M(H)\otimes R$. The quotient of this morph. is loc. isom. to $V_0 \otimes R$ over R. (V_0 corresp. to $\mathrm{Lie}(G)$)

(2) $\forall_{x=\mathrm{Spa}(K,K^+)\in \mathrm{Spa}(R,R^+)}$ the map $V \longrightarrow \tilde{H}_\eta^{ad}(K,K^+)$ is injective

Group actions on $\mathcal{M}_\infty^{EL}$:
- $G(\mathbb{Q}_p)$ acts on $\mathcal{M}_\infty^{EL}$ ($G = GL_B(V)$)
- $J(\mathbb{Q}_p) = \mathrm{End}_B(H) = \{ g \in GL_B(V) \,/\, \sigma g = b^{-1} g b \}$

Remark: (1) this avoids the use of Hecke corr.

(2) $J(\mathbb{Q}_p)$ acts on every $\mathcal{M}_n^{EL}$ and $G(\mathbb{Q}_p)$ acts only on $\mathcal{M}_\infty^{EL}$

Groth.-Mess. period map: $\pi_{GM} : \mathcal{M}_0^{EL} \longrightarrow FL_{GM}$ ← param. B-equiv. quot. of $M(H)\otimes_{W(k)} R$ which are finite proj. R-modules and locally on R are isom to $V_0 \otimes R$ as $B\otimes R$-modules

$$\begin{tikzcd}
\mathcal{M}_\infty^{EL} \arrow[d] \arrow[dd, bend right=60, "\pi_{GM}"'] & V \to \tilde{H}_\eta^{ad} \arrow[d, mapsto] & G(\mathbb{Q}_p)\text{-orbits} \arrow[dd, mapsto] \\
\mathcal{M}_0^{EL} \arrow[d, "\pi_{GM}"] & M(H)\otimes R / q\log(V) & \\
FL_{GM} & & (\text{the same point})
\end{tikzcd}$$

$FL_{GM} \circlearrowleft G(\mathbb{Q}_p) \times J(\mathbb{Q}_p)$

$\nearrow$ trivial act. $\quad \uparrow$ from its action on H

Hodge-Tate period map: We will define $\mathcal{M}_\infty^{EL} \longrightarrow FL_{HT}$
(only exists at ∞-level) Let FL_{HT} be the adic space param.

$(R,R^+) \longmapsto$ B-equiv. quotients of $V\otimes R$ which are loc. isom. to $V_1 \otimes R$ as $B\otimes R$-mod.

Prop. $\exists_{HT}$ period map $\pi_{HT} : \mathcal{M}_\infty^{EC} \to \mathcal{F}\ell_{HT}$ $G(\mathbb{Q}_p)\times J(\mathbb{Q}_p)$ - equiv.:

$\circlearrowleft_{G(\mathbb{Q}_p)\times J(\mathbb{Q}_p)} \leftarrow$ trivial action.

proof: $\forall_{x\in\mathcal{M}_\infty^{EC}} \rightsquigarrow$ p-div gp $(G,\rho)\in(\mathcal{M}^{EL,ad}_{\infty,\eta})(R,R^+)$ $\Rightarrow V \to V(G)^{ad}_\eta$

$$\begin{tikzcd}
 & V \arrow[r, "\alpha"] \arrow[dd, dotted] & V(G)^{ad}_\eta(R,R^+) \arrow[d] & & \\
 & & \tilde{H}^{ad}_\eta(R,R^+) \arrow[d, "q\log"] \arrow[dr] & & \\
0 \arrow[r] & (\mathrm{Lie}\,G^\vee)^\vee\otimes R \arrow[r] & M(H)\otimes R \arrow[r] & \mathrm{Lie}\,G\otimes R \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow$ $V\otimes R \to (\mathrm{Lie}\,G^\vee)^\vee\otimes R$

Claim: this map is surjective

Proof of claim: enough to check it on points $x = \mathrm{Spa}(K,K^+)\in\mathrm{Spa}(R,R^+)$.

The quot of $M(H)\otimes K/_{q\log(V)} \xrightarrow{\sim} V_0$ $\Rightarrow$ $\dim(q\log(V)) = h-d$ $\Rightarrow$ surjectivity

Define: $\pi_{HT}(x) :=$ quotient $V\otimes R \twoheadrightarrow (\mathrm{Lie}(G^\vee))^\vee\otimes R$ (loc. isom. to $V_1\otimes R$) □

§2. The dual data

Fact: $b\in G(W(k)[\frac{1}{p}])/\sim_\sigma$ is basic iff $\breve{B} := \mathrm{End}^0_B(H) = \mathrm{End}_B(H)\otimes\mathbb{Q}$

$\breve{B}\otimes_{\mathbb{Q}_p} W(k)[\frac{1}{p}] = \mathrm{End}_{B\otimes W(k)}(M(H)[\frac{1}{p}])$, in particular $\breve{B} = \mathrm{End}_B(V)$

Define dual data: $\breve{B} = \mathrm{End}^0_B(H) = \mathrm{End}_B(V) \supset \mathcal{O}_{\breve{B}} = \mathrm{End}_{\mathcal{O}_B}(H)$

let $\breve{V} = \breve{B} \supset \breve{\Lambda} = \mathcal{O}_{\breve{B}}$. $\breve{G} = GL_{\breve{B}}(\breve{B}) \cong J \subset \breve{B}$

$\breve{H} := \Lambda^*\otimes_{\mathcal{O}_B} H$ (i.e. $M(\breve{H}) := \Lambda^*\otimes_{\mathcal{O}_B} M(H) \circlearrowleft \mathcal{O}_{\breve{B}}$), $\Lambda^* = \mathrm{Hom}_{\mathcal{O}_B}(\Lambda,\mathcal{O}_B)$

and $\breve{\mu} : \mathbb{G}_m \to \breve{G}_{\bar{\mathbb{Q}}_p}$ $\breve{V}\otimes W(k)[\frac{1}{p}] \simeq \mathrm{End}_B(H)\otimes W(k)[\frac{1}{p}] \underset{\text{Fact}}{=} \mathrm{End}_B(M(H)[\frac{1}{p}]) =$

define $\breve{\mu}$ via: $= \mathrm{End}_B(V\otimes_{\mathbb{Q}_p} W(k)[\frac{1}{p}]) = \overline{(\mathrm{Hom}_B(V_0,V)\otimes W(k)[\frac{1}{p}])} \oplus \overline{} = \breve{V}_0$

wt 0 of $\breve{\mu}$ $\mathrm{Hom}_B(V_1,V)\otimes W(k)[\frac{1}{p}] \simeq \breve{V}_1$

wt 1 of $\breve{\mu}$

Prop. There is a natural action of G on $\breve{\mathcal{F}\ell}_{GM}$ and $\mathcal{F}\ell_{HT} \overset{G\text{-eq.}}{\cong} \breve{\mathcal{F}\ell}_{GM}$ of $\breve{G} = J$ on $\breve{\mathcal{F}\ell}_{HT}$ and $\mathcal{F}\ell_{GM} \overset{J\text{-eq.}}{\cong} \breve{\mathcal{F}\ell}_{HT}$.

proof: Recall: $\mathcal{F}\ell_{HT} \rightsquigarrow$ B-equiv. quot. of $V\otimes R$ finite. proj. R-mod loc. isom. to $V_1\otimes R$ as $B\otimes R$ modules.

$(V\otimes R \twoheadrightarrow W_1 \underset{loc}{\cong} V_1\otimes R)$

points of $\breve{\mathcal{F}\ell}_{GM} \to \underset{\mathrm{End}_B(V)}{\overset{\shortparallel}{\breve{B}}}$-equiv quot. of $M(\breve{H})\otimes R$ loc. isom. to $\mathrm{Hom}_{\breve{B}}(V_0,V)\otimes R$

$M(\breve{H})\otimes R = \breve{V}^*\otimes M(\breve{H})\otimes R = V^*\otimes V\otimes R = \mathrm{End}_B(V)\otimes R$

s.t. $\mathrm{End}_B(V)\otimes R \twoheadrightarrow \breve{W} \underset{loc.}{\cong} \mathrm{Hom}_B(V_0,V)\otimes R$.

so now we get the duality by: $0 \to \overset{\simeq V_0}{W_0} \to V\otimes R \to \overset{\simeq V_1 \otimes R}{W_1} \to 0$

apply $\mathrm{Hom}_B(-,V) \Rightarrow 0 \to \mathrm{Hom}_B(W_1,V) \to \mathrm{End}_B(V)\otimes R \to \underset{\underset{\mathrm{Hom}_B(V_0,V)}{\wr\,\mathrm{loc.}}}{\mathrm{Hom}_B(W_0,V)} \to 0$

<u>Claim.</u> this is an isom. of flag varieties.

The action of G on $\mathcal{F}\ell_{HT}$ identifies with an action on $\check{\mathcal{F}\ell}_{GM}$.

similar argument for $\mathcal{F}\ell_{GM} \underset{J\text{-equiv}}{\cong} \check{\mathcal{F}\ell}_{HT}$ $\square$

<u>Thm. (Scholze-Weinstein)</u> $\exists$ $G(\mathbb{Q}_p)\times\check{G}(\mathbb{Q}_p)$ - equiv. isom.

$$\begin{tikzcd} \mathcal{M}_\infty^{EL} \arrow[r, phantom, "\cong"] \arrow[d, "\pi_{GM}"'] & \check{\mathcal{M}}_\infty^{EL} \arrow[d, "\check{\pi}_{HT}"] \\ \mathcal{F}\ell_{GM} \arrow[r, phantom, "\cong"] & \check{\mathcal{F}\ell}_{HT} \end{tikzcd}$$

identifying π_{GM} with $\check{\pi}_{HT}$.

<u>proof:</u> (R,R^+) complete aff. $(\check{E},\mathcal{O}_{\check{E}})$-alg. Let $s\in\mathcal{M}_\infty^{EL} \leftrightarrow V \to \tilde{H}_\eta^{ad}(R,R^+)$ s.t. the quotient $M(H)\otimes R \twoheadrightarrow W \underset{loc}{\cong} V_0\otimes R$ and is injective at every (geom.) point.

$\check{s}\in\check{\mathcal{M}}_\infty^{EL} \leftrightarrow \check{V} \to \tilde{H}_\eta^{ad}(R,R^+)$ s.t. the quotient

$M(\check{H})\otimes R \twoheadrightarrow \check{W} \underset{loc}{\cong} \mathrm{Hom}_B(V_0,V)\otimes R$ and is injective at every (geom.) point.

we identify $\check{s}: \check{V} = \mathrm{End}_B(H)[\frac{1}{p}] \to V^*\otimes\tilde{H}_\eta^{ad}(R,R^+)$

Start with $s: V \to \tilde{H}_\eta^{ad}(R,R^+) \longmapsto \check{s}: \mathrm{End}_B(H) \to V^*\otimes\tilde{H}_\eta^{ad}(R,R^+)$

for $f\in\mathrm{End}_B(H) \rightsquigarrow \big\downarrow \tilde{f}_*$ $\tilde{H}_\eta^{ad}(R,R^+)$ $\qquad f \longmapsto (v \longmapsto \tilde{f}(s(v)))$

<u>claim.</u> $\check{s} \longmapsto s(v) = \check{s}(id)(v)$.

This gives a bijection

For <u>the quotient</u>: <u>claim</u>. the map induced by $\check{s}$: $\underset{\underset{\mathrm{End}_B(H)\otimes R}{\|}}{\check{V}\otimes R} \longrightarrow \underset{\underset{\mathrm{End}_B(V)}{\|}}{M(\check{H})\otimes R}$

is induced by $\mathrm{Hom}_B(-,V)$ applied to the map induced by s : $V\otimes R \to M(H)\otimes R \rightsquigarrow V\otimes R \to \underbrace{(\mathrm{Lie}\, G^\vee)^\vee}_{\cong V_1}\otimes R$

and this is compatible with period maps.

$\square$

Example: LT: $B = \mathbb{Q}_p$, H formal $\mathbb{Z}_p$-module of dim 1 and ht n.

$V = \mathbb{Q}_p^n$ $G = GL_n(\mathbb{Q}_p)$, $J = (D^{1/n}_{\mathbb{Q}_p})^\times$. $\mathcal{F}l_{GM} = \mathbb{P}^{n-1}$, $\mathcal{M}_\infty^{LT}$

Drinfeld: $\breve{B} = D^{1/n}_{\mathbb{Q}_p} = \breve{V}$, $\breve{G} = J$ $\dim \breve{V}_0 = n$, $\dim \breve{V}_1 = n^2 - n$

$\breve{H}$ = special formal $\mathcal{O}_D$-module of dim n, ht n^2.

$\breve{\mathcal{F}l}_{GM} = \Omega^{n-1} = \mathbb{P}^{n-1} \setminus \bigcup \mathbb{Q}_p$-rat. hyperplanes, $\mathcal{M}_\infty^{Dr}$.

We have

$$\begin{tikzcd}
 & \mathcal{M}_\infty^{LT} \arrow[r, "\simeq"] \arrow[d] \arrow[ddr] & \mathcal{M}_\infty^{Dr} \arrow[d] \arrow[ddl] & \\
 & \mathcal{M}^{LT,rad}_{0,\eta} = \coprod_{\mathbb{Z}} \mathring{\mathbb{B}}^{n-1} \arrow[d] & \mathcal{M}^{Dr,rad}_{0,\eta} = \coprod_{\mathbb{Z}} \Omega^{n-1} \arrow[d] & \\
D^\times \circlearrowright & \mathbb{P}^{n-1} & \Omega^{n-1} & \circlearrowleft GL_n(\mathbb{Q}_p)
\end{tikzcd}$$

Application: Thm. Let $\tilde{\Omega} \to \Omega$ be finite étale $GL_n(\mathbb{Q}_p)$-equivariant.
Then $\exists m$ s.t.

$$\begin{tikzcd}
 & \tilde{\Omega} \arrow[r] & \Omega \\
\mathcal{M}_m^{Dr} \arrow[ur, dashed] \arrow[urr] & &
\end{tikzcd}$$

.

Proof: $\widehat{\mathcal{M}}_\infty^{Dr} \underset{GL_n(\mathbb{Q}_p)\times D^\times}{\cong} \widehat{\mathcal{M}}_\infty^{LT}$

Lemma: {cat. of fin. étale adic spaces $/\mathbb{P}^{n-1}$} $\cong$
$\cong$ {$GL_n(\mathbb{Q}_p)$-equiv. finite étale $/\mathcal{M}_\infty^{LT}$}

proof: $\mathcal{M}_\infty^{LT} / GL_n(\mathbb{Q}_p) \cong \mathbb{P}^{n-1}$.

fiber product ↘

We have

$$\begin{tikzcd}
\tilde{\Omega} \arrow[dd] & \tilde{\widehat{\mathcal{M}}}{}^{Dr}_\infty \arrow[l] \arrow[d] & = \tilde{\widehat{\mathcal{M}}}{}^{LT}_\infty \arrow[d, "GL_n(\mathbb{Q}_p)\text{-eq.}"] \\
 & \widehat{\mathcal{M}}^{Dr}_\infty \arrow[dl] & = \mathcal{M}^{LT}_\infty \\
\Omega & &
\end{tikzcd}$$

By rigid GAGA, any finite étale map over $\mathbb{P}^{n-1}$ is algebraic and split hence

$$\begin{tikzcd}
\widehat{\mathcal{M}}^{Dr}_\infty \arrow[r] \arrow[dr, "?"'] & \tilde{\widehat{\mathcal{M}}}{}^{Dr}_\infty \arrow[r] & \tilde{\Omega} \\
 & \mathcal{M}^{Dr}_\infty \arrow[ur, dashed, "\text{Claim}"'] &
\end{tikzcd}$$

have to prove the claim.

□