Local Langlands for GL_n. (Konstantin Ardakov)

Representations of $GL_n(F)$.

$F/\mathbb{Q}_p$ finite extension
$\mathcal{O} \subseteq F$ ring of integers, $\pi \in \mathcal{O}$ uniformiser, $k = \mathcal{O}/\pi\mathcal{O}$.

- Interested in $\rho : \underbrace{GL_n(F)}_{=:G} \to End(V)$,
  - V a vector space/$\mathbb{C}$.
  - $\forall v \in V \setminus \{0\}$, $Stab_G(v)$ is open in G. (smoothness)
  - V is admissible iff $\dim V^K < \infty$ for all open compact $K \leq G$.

We're interested in irreducible smooth admissible V as above.

- If $n_1 + \cdots + n_r = n$ (partition into positive integers) & π_i are smooth adm. repns of $GL_{n_i}(F)$, can form $Ind_{P_{\underline{n}}}^G(\pi_1 \times \pi_2 \times \dots \times \pi_r)$: this is again sm. adm.
- If π arises as a subrepn of one of these (with $r \geq 2$), then π is not supercuspidal, otherwise π (irred, sm. adm.) is supercuspidal.

Defn. $\mathcal{A}_n(F) := \{\text{irr. sm. adm. } GL_n(F)\text{-repns}\}/\sim$
$\cup\!|$
$\mathcal{A}_n^0(F) := \{\text{supercuspidal } \pi\}/\sim$.

Thm. (Bernstein & Zelevinsky(?)) Every $\pi \in \mathcal{A}_n(F)$ appears as a subquotient of some $Ind_{P_{\underline{n}}}^G(\pi_1 \times \dots \times \pi_r)$, with $\pi_i \in \mathcal{A}_{n_i}^0(F)$.

Weil–Deligne representations.

$$G_F = \underset{\text{(prosoluble)}}{\mathrm{Gal}(\overline{F}/F)} \xrightarrow{\ -\ } G_k = \mathrm{Gal}(\overline{k}/k)$$

$I_F = \ker(-)$ inertia gp. Has the structure:

$$1 \to P_F \to I_F \to \prod_{\ell \neq p} \mathbb{Z}_\ell \to 1 \qquad \text{(s.e.s.)}$$

where P_F is pro-p.

We're interested in $\{$f.d. repns $G_F \to GL_n(\overline{\mathbb{Q}}_\ell)\}/\sim$ $(\ell \neq p)$

(arise as étale coh. gps of geometric spaces)

Weil group. W_F is defined by exactness of

$$\begin{tikzcd}
1 \arrow[r] & I_F \arrow[r] \arrow[d, equal] & W_F \arrow[r, "\nu"] \arrow[d, hook] & \mathbb{Z} \ni 1 \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & I_F \arrow[r] & G_F \arrow[r] & G_k \ni \mathrm{Frob}. \arrow[r] & 1
\end{tikzcd}$$

- I_F is open in W_F.

Defn. A Weil–Deligne repn is a ~~smooth~~ pair (ρ, N), where ρ is a smooth repn $W_K \to GL_n(\overline{\mathbb{Q}}_\ell)$ and $N \in M_n(\overline{\mathbb{Q}}_\ell)$ is nilpotent, and

$$\rho(w) N \rho(w)^{-1} = \|w\|^{-1} N, \qquad \|w\| = |k|^{-\nu(w)}.$$

- (ρ, N) is Frobenius-semisimple if ρ is a s.s. $\overline{\mathbb{Q}}_\ell[W_F]$-module.

(any lift of Frob. must act semisimply)

~~Thm (Harris–Taylor, Henn~~

$$\mathcal{G}_n(F) = \{\text{Weil-Deligne repns } (\rho, N)\}/\sim$$
$$\cup$$
$$\mathcal{G}_n^0(F) = \{(\rho, 0) : \rho \text{ is irred.}\}/\sim$$

<u>Thm.</u> (Harris–Taylor, Henniart) (LLC for $GL_n(F)$)
There is a unique set of bijections

$$\{rec_n \mid n \geq 1\}, \text{ where } rec_n : \mathcal{A}_n(F) \longrightarrow \mathcal{G}_n(F)$$

s.t.
- $rec_n(\mathcal{A}_n^0(F)) = \mathcal{G}_n^0(F)$
- rec_1 is induced from the Local Class Field Theory isomorphism $W_F^{ab} \xrightarrow{\sim} F^\times = GL_1(F)$.
  $Frob^{-1} \longleftrightarrow \pi$
- rec_n are compatible with:
  - L-factors, ε-factors, duals, conductors.

Sketch of construction used in [H–T].

Let $\mathbb{X}$ be a <u>formal $\mathcal{O}$-module</u> over $\bar{k}$:

($\mathbb{X}$ is a 1-d. formal group $/\bar{k}$ + an action of $\mathcal{O}$, i.e. a ring hom $\mathcal{O} \longrightarrow End_{\bar{k}}(\mathbb{X})$, + ... (compatibility))

Say that $\mathbb{X}$ has <u>F-height</u> n if $\ker(\pi_{\mathbb{X}})$ is a finite groupscheme over $\bar{k}$, of rank $|k|^n$.

<u>Thm.</u> (Drinfeld.) $\mathbb{X}$ is unique up to isomorphism (in the cat. of formal $\mathcal{O}$-mods$/\bar{k}$).

Drinfeld considers deformations of $\mathbb{X}$ to
$\mathcal{C} := \{$complete local noetherian $\mathcal{O}$-algs$\}$.

Thm. (Drinfeld.) Let B be the division algebra of dimension n^2 over F, with invariant $\frac{1}{n}$. Then $B \cong \mathrm{End}_k(\mathbb{X})$.

A deformation of $\mathbb{X}$ over $R \in \mathcal{C}$ is a pair (X, ι) where X is a formal $\mathcal{O}$-module$/R$ and
$\iota : \mathbb{X} \longrightarrow X \times_k k$ is an isomorphism.

Defn. $\mathcal{M}_0(R) = \{(X,\iota)\}/\sim$ functor on $\mathcal{C}$.

Thm. (Drinfeld.) $\mathcal{M}_0$ is (pro)represented by

$\mathcal{A} := \widehat{\bar{\mathcal{O}}}[[u_1, \dots, u_{n-1}]]$ ($\bar{\mathcal{O}}$ is the ring of integers in F^{ur}, the maximal unramified extension of F)

- $\mathrm{Spf}(\mathcal{A})$ is a point.
- Berthelot–Raynaud defined a generic fibre
M_0: $\mathrm{Sp}(\mathcal{A}_{\widehat{F^{ur}}})$ – this is a rigid analytic space over $\widehat{F^{ur}}$, ~~these~~ isomorphic to an open ~~disc~~ polydisc "$\{z \in \widehat{F^{ur}}^{n} \mid |z| < 1\}$".

Now the idea is to consider

$$H^i_c(M_{0,ét}, \overline{\mathbb{Q}}_\ell) \qquad (c:\ \text{compactly supported})$$

– this admits commuting actions of $I_F \cong G_{\widehat{F^{ur}}}$ and $\mathcal{O}_B^\times$.
(connected cpt of base space of Lubin–Tate tower)

~~(Also $Gl_n(\mathcal{O})$ is acting on this(?).)~~

Now let $LT_0 := \coprod_{i \in \mathbb{Z}} M_0$. (non-qcpt rigid analytic space)

Thm. (Drinfeld.) There is a tower $\{LT_m\}_{m \geq 0}$

$$\cdots \to LT_3 \to LT_2 \to LT_1 \to LT_0$$

s.t.
- LT_m is finite étale $/ LT_0$
- $Gal(LT_m / LT_0) \cong Gl_n(\mathcal{O}/\pi^m \mathcal{O})$.

(LT_m is the generic fibre of a formal scheme that is the universal deformation space of (X, ι, ϕ), where ϕ is a ~~local~~ Drinfeld level m structure.)

Def. Let $\mathcal{H}^i := \varinjlim_m H^i_c(LT_m, \overline{\mathbb{Q}}_\ell)$.

Thm. (Carayol(?), Berkovich)
- the action of $I_F \times \mathcal{O}_B^\times$ on $H^i_c(LT_0, \overline{\mathbb{Q}}_\ell)$ extends to an action of (a subgroup of ?) $W_F \times B^\times \times Gl_n(F)$ on $\mathcal{H}^i$.
  $\{(w, b, g)$ s.t. $|\det g| = |N(b)| \cdot \|w\|$, $N : B^\times \to F^\times\}$
- this representation is smooth.

Thm. (Harris–Taylor.) If $\pi \in \mathcal{A}^0_n(F)$, then ~~$\exists r(\pi)$~~ $\exists\, r(\pi) : W_F \to Gl_n(\overline{\mathbb{Q}}_\ell)$ s.t.

$$\sum_{i \in \mathbb{Z}} (-1)^i \left[\mathrm{Hom}_{Gl_n(F)}(\pi, \mathcal{H}^i)\right] = \left[JL(\pi) \otimes r(\pi)\right]$$

in $K_0(\{\text{f.d. smooth } B^\times \times W_F\text{-reps}/\overline{\mathbb{Q}}_\ell\})$.

<u>Scholze</u>: similar strategy, but different in the following respects:

— he gets $\overset{\sigma}{\cancel{\mathrm{rec}_n}} : \mathcal{A}_n(F) \longrightarrow \left\{ \begin{matrix} n\text{-dim}^{l} \text{ s.s.} \\ W_F\text{-repns} \end{matrix} \right\}$

s.t. $\overset{\sigma}{\cancel{\mathrm{rec}_n}}$ agrees with H–T's rec_n on $\mathcal{A}_n^0(F)$.

If π is any subquotient of $\mathrm{Ind}_{P_{\underline{n}}}^{GL_n(F)}(\pi_1 \times \cdots \pi_r)$ with $\pi_i \in \mathcal{A}_{n_i}^0(F)$, then

$$\sigma(\pi) = \sigma(\pi_1) \oplus \cdots \oplus \sigma(\pi_r).$$

— uses moduli spaces of p-divisible groups, instead of formal $\mathcal{O}$-modules.

— no action of $B^\times$.